Find The Jacobian Of The Transformation X=u, Y=2uv And Sketch The Region G: 1u2, 22uv4, In The Uv-plane.

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Understanding the Jacobian of a transformation is fundamental in multivariable calculus, especially when changing variables in multiple integrals. In this article, we will explore how to find the Jacobian for the given transformation and analyze the region defined in the uv-plane. This process involves calculating the Jacobian determinant, which accounts for how areas are scaled under the transformation, and sketching the region G to visualize the domain of integration.

Introduction to Transformation and Jacobian

Before diving into the specific problem, it is essential to grasp the concepts involved—namely, transformations between coordinate systems and the Jacobian determinant.

What is a Transformation?

A transformation maps points from one coordinate system to another. In our case, the transformation is defined by:

- $X = u$
- $Y = 2uv$

This maps the (u, v) plane into the (X, Y) plane. Such transformations are common in calculus when simplifying integrals or analyzing regions.

Understanding the Jacobian Determinant

The Jacobian determinant measures how area elements change when transforming between coordinate systems. If $\mathbf{T}(u, v) = (X(u, v), Y(u, v))$, then the Jacobian J is given by:

\[
J = \det \begin{bmatrix}

\frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\
\frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v}
\end{bmatrix}
\]

The magnitude of J indicates how areas in the uv-plane are scaled when mapped to the XY-plane.

Step-by-Step Calculation of the Jacobian

Let's now focus on calculating the Jacobian for the specific transformation:

- $X = u$
- $Y = 2uv$

Partial Derivatives

Compute the partial derivatives:

$$
\frac{\partial X}{\partial u} = 1, \quad \frac{\partial X}{\partial v} = 0
$$
$$
\frac{\partial Y}{\partial u} = 2v, \quad \frac{\partial Y}{\partial v} = 2u
$$

Jacobian Determinant

The Jacobian determinant is:

$$
J = \det \begin{bmatrix}
1 & 0 \\
2v & 2u
\end{bmatrix} = (1)(2u) - (0)(2v) = 2u
$$

Therefore, the Jacobian of the transformation is:

$$
\boxed{J = 2u}
$$

This result indicates that the area element in the XY-plane relates to the uv-plane by a factor of $2u$.

Analyzing the Region G in the uv-plane

The problem also asks us to analyze and sketch the region G defined by the inequalities:

- $1u2$, which we interpret as $1 \le u \le 2$
- $22uv4$, which we interpret as $2 \le u v \le 4$

Let's clarify these conditions and understand their implications.

Interpreting the Region G

Given the inequalities:

- $1 \le u \le 2$
- $2 \le u v \le 4$

we can analyze the region in the uv-plane accordingly.

Expressing v in terms of u

From the inequalities involving $u v$:

$$
2 \le u v \le 4
$$

Dividing through by u (which is positive in the region $u \in [1,2]$):

$$
\frac{2}{u} \le v \le \frac{4}{u}
$$

Thus, for each fixed u in [1, 2], the corresponding v varies between $\frac{2}{u}$ and $\frac{4}{u}$.

Sketching the Region G

To visualize the region:

1. For u in [1, 2], the vertical bounds are:

$$
v = \frac{2}{u} \quad \text{and} \quad v = \frac{4}{u}

\]

2. When \(u=1\):

\[
v \in [2, 4]
\]

3. When \(u=2\):

\[
v \in [1, 2]
\]

4. The curves \(v = 2/u\) and \(v=4/u\) are rectangular hyperbolas, with the first decreasing as \(u\) increases.

Plotting these:

- The lower boundary: \(v=2/u\)
- The upper boundary: \(v=4/u\)
- The vertical limits: \(u = 1\) and \(u=2\)

The region \(G\) is bounded between these two hyperbolas over \(u \in [1, 2]\).

Summary of the Region G

The region \(G\) in the uv-plane can be described as:

\[
G = \left\{ (u, v) \mid u \in [1, 2], \quad \frac{2}{u} \le v \le \frac{4}{u} \right\}
\]

This region is a "band" between two hyperbolic curves, spanning \(u\) from 1 to 2.

Applications of the Jacobian in Integration

Once the Jacobian is known, it can be used to evaluate double integrals over the region \(G\) or its image in the XY-plane.

Changing Variables in Integrals

Suppose we want to evaluate an integral of a function \(f(x, y)\) over a region \(R\) in the XY-plane. Using the transformation \(X = u\), \(Y=2uv\), the integral becomes:

$$
\iint_{R} f(x, y) \, dx\,dy = \iint_{G} f(u, 2uv) \, |J| \, du\,dv
$$

where $|J| = |2u|$ is the absolute value of the Jacobian determinant.

Implication of the Jacobian Sign

Since $u \in [1, 2]$, the Jacobian $J=2u$ is always positive, so $|J|=2u$. This simplifies calculations and ensures correct area scaling during the transformation.

Conclusion and Summary

In this comprehensive exploration, we:

- Calculated the Jacobian determinant for the transformation $X = u$, $Y=2uv$, obtaining $J = 2u$.
- Analyzed and sketched the region G in the uv-plane, bounded between hyperbolic curves $v=2/u$ and $v=4/u$, for u in [1, 2].
- Discussed the importance of the Jacobian in changing variables for multiple integrals and how it relates the area elements between the uv-plane and the XY-plane.

This understanding is crucial for solving complex integrals involving coordinate transformations, especially in applications like physics, engineering, and advanced calculus. By mastering the calculation of the Jacobian and the geometric interpretation of regions, students and professionals can approach multivariable calculus problems with greater confidence and clarity.

Additional Tips for Practice

- Always verify the domain of your variables before computing the Jacobian, especially when inequalities are involved.
- When sketching regions defined by inequalities involving hyperbolas or other curves, plot key points (like where $u=1, 2$) to understand the shape.
- Remember that the Jacobian's sign indicates orientation; for area calculations, use the absolute value.

Mastering these concepts enhances your ability to work with multivariable transformations efficiently and accurately.

Frequently Asked Questions

What is the transformation from (u, v) to (x, y) given by X = u and Y = 2uv?

The transformation maps (u, v) to (x, y) where $x = u$ and $y = 2uv$.

How do you find the Jacobian of the transformation X = u, Y = 2uv?

The Jacobian is found by computing the determinant of the matrix of partial derivatives: $J = |\partial(x, y)/\partial(u, v)|$.

What is the Jacobian determinant for the given transformation?

The Jacobian determinant is $|\partial x/\partial u \ \partial x/\partial v; \partial y/\partial u \ \partial y/\partial v| = |1 \ 0; 2v \ 2u|$, which simplifies to $2u$.

How do you compute the partial derivatives for the Jacobian in this case?

Since $x = u$, $\partial x/\partial u = 1$ and $\partial x/\partial v = 0$; for $y = 2uv$, $\partial y/\partial u = 2v$ and $\partial y/\partial v = 2u$.

What are the bounds for the region G in the (u, v) plane?

The region G is defined by $1 \leq u \leq 2$ and $2 \leq uv \leq 4$.

How do you sketch the region G in the (u, v) plane based on the given bounds?

Plot the lines $u = 1$ and $u = 2$, and for each, find v such that $2 \leq uv \leq 4$, which gives v between $2/u$ and $4/u$, resulting in a bounded region between these curves.

How does the Jacobian affect the change of variables when integrating over region G?

The Jacobian determinant converts area elements from the (u, v) plane to the (x, y) plane, so $dx\,dy = |J|\,du\,dv$, accounting for the area distortion.

What is the final expression for the Jacobian when transforming from (u, v) to (x, y)?

The Jacobian is $|J| = 2u$, which multiplies the differential area element du dv in the integral.

Why is understanding this Jacobian important for evaluating integrals over the transformed region?

Knowing the Jacobian ensures accurate computation of integrals after change of variables, accounting for how areas are scaled under the transformation.

Additional Resources

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Introduction

Understanding how different coordinate systems relate and transform is fundamental in advanced calculus and multivariable analysis. Transformations allow us to simplify complex regions, evaluate integrals more easily, and analyze functions in more convenient coordinate spaces. One such transformation, which often appears in applications ranging from physics to engineering, involves mapping from the uv-plane to the XY-plane through specific algebraic relations. Today, we'll explore the transformation defined by:

- \(X = u \)
- \(Y = 2uv \)

Our goal is twofold: first, to compute the Jacobian determinant of this transformation, a crucial step in changing variables in multiple integrals; second, to sketch the region \(G \) in the uv-plane described by the inequalities:

- \(1 \leq u \leq 2 \)
- \(2 \leq uv \leq 4 \)

By doing so, we will develop a comprehensive understanding of how the original region maps into the uv-plane and how to utilize the Jacobian in practical calculations.

Understanding the Transformation

Before delving into the Jacobian, it's important to grasp the nature of the transformation itself.

The Transformation Equations

Given:
\[
X = u
\]
\[

Y = 2uv
\]

This transformation maps a point \((u, v)\) in the uv-plane to a point \((X, Y)\) in the XY-plane. Notice that:

- The \(X\)-coordinate is directly the variable \(u\).
- The \(Y\)-coordinate depends on the product \(uv\), scaled by 2.

This type of transformation is common in scenarios where the variables are multiplicatively related, such as in polar-like coordinate systems or in certain physics problems involving areas and flux.

Computing the Jacobian Determinant

The Jacobian determinant measures how an infinitesimal area element in the uv-plane transforms into an area element in the XY-plane. It is a vital component in changing variables in double integrals, ensuring the correct scaling is applied.

Step 1: Recall the Jacobian Formula

For a transformation:
\[
X = X(u, v), \quad Y = Y(u, v),
\]
the Jacobian determinant \(J\) is given by:
\[
J = \left| \frac{\partial (X, Y)}{\partial (u, v)} \right| =
\begin{vmatrix}
\frac{\partial X}{\partial u} & \frac{\partial X}{\partial v} \\
\frac{\partial Y}{\partial u} & \frac{\partial Y}{\partial v}
\end{vmatrix}
\]

Step 2: Compute the Partial Derivatives

Based on the transformation:

- \(X = u\)

\[
\frac{\partial X}{\partial u} = 1
\]
\[
\frac{\partial X}{\partial v} = 0
\]

- \(Y = 2uv\)

$$
\frac{\partial Y}{\partial u} = 2v
$$

$$
\frac{\partial Y}{\partial v} = 2u
$$

Step 3: Calculate the Determinant

Putting these derivatives into the Jacobian matrix:

$$
J =
\begin{vmatrix}
1 & 0 \\
2v & 2u
\end{vmatrix}
= (1)(2u) - (0)(2v) = 2u
$$

Thus, the Jacobian determinant is:

$$
\boxed{
J = 2u
}
$$

This expression indicates that the local area element in the uv-plane scales by a factor of $2u$ when mapped into the XY-plane. Notably, since u is constrained to the interval $[1, 2]$, the Jacobian is always positive and bounded between 2 and 4, which simplifies analysis.

Sketching the Region G in the uv-plane

The second part of our exploration involves understanding the region G, defined by the inequalities:

- $1 \leq u \leq 2$
- $2 \leq uv \leq 4$

This region resides in the uv-plane, and visualizing it helps in setting up integrals and understanding the transformation's behavior.

Step 1: Express the Inequalities Clearly

The inequalities specify:

1. u is between 1 and 2.

2. The product uv is between 2 and 4.

Since u is bounded, we can analyze the bounds for v based on u:

$$
2 \leq uv \leq 4
$$

Dividing through by u (which is positive in our domain):

$$
\frac{2}{u} \leq v \leq \frac{4}{u}
$$

Thus, for each fixed $u \in [1, 2]$, v ranges between $\frac{2}{u}$ and $\frac{4}{u}$.

Step 2: Visualizing the Region

To sketch G:

- Draw the u-axis from 1 to 2.
- For each u in this interval, draw the vertical segment between $v = \frac{2}{u}$ and $v = \frac{4}{u}$.

Plotting these bounds:

- At $u = 1$:

$$
v \in [2, 4]
$$

- At $u = 2$:

$$
v \in [1, 2]
$$

- For u in between:

$$
v \text{ varies between } \frac{2}{u} \text{ and } \frac{4}{u}
$$

The region G is thus bounded by the curves:

- $v = \frac{2}{u}$ (lower boundary)
- $v = \frac{4}{u}$ (upper boundary)

and the vertical lines at $u=1$ and $u=2$.

This results in a "curvilinear strip" in the uv-plane, with the region narrowing as u increases due to the decreasing nature of $\frac{2}{u}$ and $\frac{4}{u}$.

Mathematical Description of Region G

Summarizing, the region G in the uv-plane can be described as:

$$
G = \left\{ (u, v) \ \middle| \ u \in [1, 2], \ \frac{2}{u} \leq v \leq \frac{4}{u} \right\}
$$

This explicit description enables us to set up double integrals over G, converting them into integrals over the u and v variables with the Jacobian included.

Application: Changing Variables in Integrals

One common application of the Jacobian and region sketching is in evaluating double integrals where the integrand or the region is complicated in the xy-plane but simpler in the uv-plane.

Suppose we need to evaluate an integral:

$$
\iint_{R} f(x, y) dx dy,
$$

where R is the image of G under the transformation.

Using the change of variables:

$$
x = u, \quad y = 2 u v,
$$

the integral becomes:

$$
\iint_{G} f(u, 2 u v) \left| J \right| du dv
$$
$$
= \iint_{G} f(u, 2 u v) \times 2 u \, du dv
$$

By exploiting the bounds of G, the integral simplifies to:

$$
\int_{u=1}^{2} \int_{v=2/u}^{4/u} f(u, 2 u v) \times 2 u \, dv du
$$

\]

This setup demonstrates the utility of the Jacobian determinant and the explicit description of the region \(G \).

Final Remarks and Insights

- Significance of the Jacobian: The Jacobian determinant \(2u \) indicates how infinitesimal areas scale when transforming from the uv-plane to the XY-plane. Its positivity within the domain ensures the orientation is preserved, simplifying calculations.

- Region \(G \) Geometry: The inequalities defining \(G \) create a region bounded by hyperbolic curves \(v = 2/u \) and \(v = 4/u \), with \(u \) constrained to \([1, 2]\). Visualizing this helps in setting up integrals and understanding the transformation's effects.

- Practical Applications: Transformations like these are pivotal in solving complex integrals, especially when the regions have curved boundaries. Recognizing the bounds in the uv-plane simplifies the integration process, often transforming complicated regions in the xy-plane into manageable rectangles or strips.

- Broader Implications: Mastery of Jacobians and region transformations is fundamental in multiple fields such as physics (e.g., change of variables in flux calculations),

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