Find The Volume Of The Described Solid Of Revolution Or State That It Does Not Exist. The Region Bounded

Find The Volume Of The Described Solid Of Revolution Or State That It Does Not Exist. The Region Bounded is a fundamental problem in calculus, particularly in the study of solids of revolution. This task involves determining the volume of a three-dimensional object generated when a specific region in the xy-plane is revolved around a particular axis. The process combines geometric intuition with integral calculus techniques, primarily the disk, washer, and shell methods. In this comprehensive guide, we'll explore how to approach such problems, understand the underlying concepts, and provide step-by-step instructions to compute the volume or establish its non-existence.

Understanding the Basics of Solids of Revolution

Before diving into specific problem-solving strategies, it's essential to grasp the fundamental concepts involved in solids of revolution.

What Is a Solid of Revolution?

A solid of revolution is a three-dimensional shape created when a two-dimensional region in the plane is rotated around a fixed line (called the axis of revolution). Examples include:
- A sphere generated by revolving a semicircle around its diameter.
- A torus formed by revolving a circle around an external axis.

Common Axes of Revolution

- x-axis: The horizontal axis in the xy-plane.
- y-axis: The vertical axis.
- Vertical or horizontal lines: Any other line parallel to the axes, such as x = a or y = b.

Revolution Methods

The primary calculus methods used to find volumes of revolution are:
- Disk Method: Suitable when the region is bounded by functions of x or y,

and the axis of revolution is perpendicular to the coordinate axis.
- Washer Method: An extension of the disk method, used when the region is between two curves, creating a "hole" in the solid.
- Shell Method: Useful when revolving around the y-axis or other vertical lines, especially when the region is better described in terms of x.

Step-by-Step Approach to Find the Volume

To accurately determine the volume of a solid of revolution, follow these structured steps:

1. Understand the Region and Its Boundaries

- Identify the region in the xy-plane bounded by given curves, lines, or points.
- Sketch the region to visualize the shape and the axis of rotation.

2. Determine the Axis of Revolution

- Clarify whether the region is being revolved around the x-axis, y-axis, or an arbitrary line.
- This choice influences the method used (disk, washer, shell).

3. Decide the Appropriate Method

- Use the disk/washer method if the cross-sections perpendicular to the axis are disks or washers.
- Use the shell method if the shells formed by revolving vertical or horizontal strips are easier to integrate.

4. Set Up the Integral Expression

- For disks: $V = \pi \int_{a}^{b} [R(x)]^2 dx$
- For washers: $V = \pi \int_{a}^{b} [R_{outer}(x)]^2 - [R_{inner}(x)]^2 dx$
- For shells: $V = 2\pi \int_{a}^{b} \text{(radius)} \times \text{(height)} dx$

5. Determine the Limits of Integration

- Find the points of intersection of the bounding curves.
- Use these points as the bounds for your integral.

6. Evaluate the Integral

- Carry out the integration carefully, simplifying as needed.
- Check for convergence; if the integral diverges, the volume does not exist.

7. Interpret the Result

- Confirm the physical validity of the volume.
- If the integral diverges or the region is unbounded, state that the volume does not exist.

Common Scenarios and Examples

To solidify understanding, let's explore typical problems involving the volume of a solid of revolution.

Example 1: Revolving a Bounded Region Around the x-Axis

Problem Statement:
Find the volume of the solid obtained by revolving the region bounded by $y = x^2$, $y = 4$, and $x = 0$ around the x-axis.

Solution Approach:
- The region is between $y = x^2$ and $y=4$, from $x=0$ to $x=2$ (since $y=4 \Rightarrow x= \pm 2$, but considering $x \geq 0$).
- The axis of revolution is the x-axis, so the disk method is suitable.

Setup:
- Outer radius: $R_{outer} = y = 4$
- Inner radius: $R_{inner} = y = x^2$ (but since the region is between these curves, the washer method applies).

Integral:
$$
V = \pi \int_{x=0}^{x=2} \left[(4)^2 - (x^2)^2 \right] dx = \pi \int_{0}^{2} (16 - x^4) dx
$$
Calculate:
$$
V = \pi \left[16x - \frac{x^5}{5} \right]_0^2 = \pi \left(16 \times 2 - \frac{32}{5} \right) = \pi \left(32 - \frac{32}{5} \right) = \pi \left(\frac{160}{5} - \frac{32}{5} \right) = \pi \frac{128}{5} = \frac{128\pi}{5}
$$

Result:
The volume is \(\boxed{\frac{128\pi}{5}} \).

Example 2: Revolving a Region Around the y-Axis

Problem Statement:
Find the volume generated by revolving the region bounded by \(y = \sqrt{x} \), \(y=0 \), and \(x=4 \) around the y-axis.

Solution Approach:
- Because the region is bounded by \(y = \sqrt{x} \), \(y=0 \), and \(x=4 \), the limits in y are from 0 to 2 (since \(x=4 \Rightarrow y=\sqrt{4} = 2 \)).
- Shell method is preferable here because revolving around the y-axis, and the region is described in x.

Setup:
- Radius of a shell at height y: \(x = y^2 \)
- Height of shell: \(\Delta x \) or the differential element \(dx \)

Integral:
\[
V = 2\pi \int_{y=0}^{2} y \times x(y) dy = 2\pi \int_0^2 y \times y^2 dy = 2\pi \int_0^2 y^3 dy
\]
Calculate:
\[
V= 2\pi \left[\frac{y^4}{4} \right]_0^2 = 2\pi \times \frac{16}{4} = 2\pi \times 4 = 8\pi
\]

Result:
The volume is \(\boxed{8\pi} \).

When Does the Volume Not Exist?

Some problems may involve unbounded regions or functions that lead to divergent integrals. In such cases, the volume of the solid of revolution may not exist.

Situations include:
- The region extends infinitely in at least one direction.
- The integral for the volume diverges (i.e., approaches infinity).

- The region is not well-defined or the functions involved are discontinuous in the interval of integration.

Example:
Revolving the region between $y = \frac{1}{x}$ and the y-axis, from $x=1$ to infinity, around the x-axis:

$$
V = \pi \int_{x=1}^{\infty} \left(\frac{1}{x} \right)^2 dx = \pi \int_{1}^{\infty} \frac{1}{x^2} dx
$$

This integral converges:

$$
V= \pi \left[-\frac{1}{x} \right]_1^{\infty} = \pi (0 + 1) = \pi
$$

Thus, the volume exists and equals π.

Conversely, if the integral diverges (e.g., $\int_{1}^{\infty} \frac{1}{x} dx$), the volume does not exist.

Summary of Key Points

- Correctly identifying the bounded region and the axis of revolution is crucial.
- Choose the appropriate method (disk, washer, shell) based on the shape and orientation.
- Set up the integral carefully, determining limits of integration from the region’s boundaries.
- Always check whether the integral converges or diverges; divergence implies the volume does not exist.
- Visualize the problem through sketches to aid in understanding and setting up the integral.

Final Tips for

Frequently Asked Questions

How do I determine the volume of a solid of

revolution formed by revolving a region bounded by curves around the x-axis?

You can use the disk or washer method by integrating π times the square of the radius function with respect to x over the given interval.

What should I do if the region is bounded by two curves and I need to find the volume of the solid of revolution around the y-axis?

Use the shell method, which involves integrating 2π times the radius times the height of the shell, with respect to y.

How can I verify if the volume of a solid of revolution exists for a given region?

Check if the defining functions are continuous and integrable over the interval; if the integral converges, the volume exists.

What are common mistakes to avoid when calculating the volume of a revolution of a bounded region?

Mistakes include mixing up the limits of integration, using the wrong method (disk vs.

shell), and neglecting to square the radius or height functions.

When the region is bounded by curves y=f(x) and y=g(x), how do I set up the integral for the volume of the revolution around the x-axis?

Set up the integral as $V = \pi\int$ from a to b of $[f(x)]^2 - [g(x)]^2\,dx$, assuming the region is between these curves and revolved around the x-axis.

Can the volume of a solid of revolution be infinite? How do I determine this?

Yes, if the region extends infinitely and the integral diverges, the volume is infinite. Check the limits and integrand behavior at infinity to determine convergence.

What does it mean when the problem states 'or state that it does not exist' regarding the volume of a solid of revolution?

It indicates that the region may not produce a finite volume, possibly because the integral diverges or the region is unbounded in a way that the volume cannot be finite.

How do I find the volume when the region is bounded by a circle and a line, revolving around the y-axis?

Identify the intersection points, express the region as functions y=f(x) or x=g(y), and use the shell method or disk method accordingly to set up the integral.

What techniques are useful if the region is irregular or bounded by complex curves when calculating the volume of revolution?

Parametrize the curves if possible, split the region into simpler parts, and apply the appropriate method (disk, washer, or shell) to each before summing the volumes.

Additional Resources

Find The Volume Of The Described Solid Of Revolution Or State That It Does Not Exist. The Region Bounded

Understanding the volume of solids formed by revolution is a fundamental concept in calculus, bridging the gap between geometric intuition and analytical precision. This technique, often called the method of disks, washers, or shells, allows mathematicians and engineers to determine the capacity of objects created when a region in the plane is revolved around an axis. Whether it’s calculating the volume of a wine bottle, a vase, or a mechanical part, these methods provide robust tools for real-world applications. This article delves into the core principles behind finding the volume of such solids, the criteria for their existence, and how to approach problems involving complex regions.

The Foundation: Understanding Regions and Axes of Revolution

Before tackling the computation of volume, it’s crucial to understand the basic elements involved:

1. The Region in the Plane

A region in the xy-plane is typically described by inequalities involving functions, such as:

- $y = f(x)$,
- $y = g(x)$,

- or by geometric boundaries, like circles, lines, and polygons.

The shape of this region—whether it's bounded above and below by functions, or enclosed within curves—directly influences the resulting solid of revolution.

2. The Axis of Revolution

The axis about which the region is revolved significantly impacts the shape and the method used:

- Horizontal axis (e.g., x-axis): revolutions generate objects like tori or shells around the x-axis.
- Vertical axis (e.g., y-axis): revolutions produce different geometries, such as cylinders or disks about the y-axis.
- Oblique axes: more complex but manageable with coordinate transformations.

The position of the region relative to the axis determines whether the volume can be finite or infinite, and whether the integral converges.

Methods for Computing Volume of Solids of

Revolution

Two primary techniques are used to find the volume:

1. Disk/Washer Method

This approach involves slicing the solid perpendicular to the axis of revolution and summing the volumes of these disks or washers.

When to use:

- The region is described by a function $y = f(x)$ between $x=a$ and $x=b$.
- The axis of revolution is horizontal (e.g., x-axis).

Formula:

For a region bounded between $y = f(x)$ and $y = g(x)$, revolved around the x-axis:

$$
V = \pi \int_{a}^{b} [f(x)]^2 - [g(x)]^2 \, dx
$$

This computes the volume of the disks or washers, depending on whether there's an inner radius (hole) or not.

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2. Shell Method

This technique involves slicing the solid parallel to the axis of revolution and summing the shells.

When to use:

- The region is described by $y = f(x)$, and the axis of revolution is vertical.
- It simplifies cases where the disk method is complicated due to the geometry.

Formula:

Revolving around the y-axis:

$$
V = 2\pi \int_{a}^{b} x \cdot f(x) \, dx
$$

The shell method often simplifies calculations, especially when dealing with functions that are easier to integrate in terms of x rather than y.

Determining the Existence of the Solid

Not all regions, when revolved, produce finite, well-defined solids. Several criteria influence whether the volume exists and is finite:

1. Boundedness of the Region

- The region must be bounded in the plane to produce a finite volume when revolved.
- Unbounded regions (e.g., extending infinitely) often lead to infinite volumes unless the shape decays rapidly enough for the integral to converge.

2. Behavior of the Functions

- Functions that tend to infinity or have discontinuities at the boundaries can cause the integral to diverge.
- For example, revolving a region near a vertical asymptote may lead to an infinite volume.

3. The Nature of the Axis

- If the region approaches the axis of revolution too closely or extends infinitely, the resulting solid may be infinite in size or non-existent in a finite sense.

Step-by-Step Approach to Find the Volume

When confronted with a problem involving a bounded region, follow these systematic steps:

Step 1: Precisely Define the Region

- Write down the inequalities or equations describing the boundaries.
- Identify the points of intersection, which determine limits of integration.

Step 2: Determine the Axis of Revolution

- Decide whether the axis is horizontal or vertical.
- Visualize or sketch the region and the axis to understand the shape.

Step 3: Choose the Appropriate Method

- Use the disk/washer method if the slices are perpendicular to the axis.
- Use the shell method if the slices are parallel to the axis or the geometry suggests it.

Step 4: Set Up the Integral

- Express the radius(s) or height(s) of the slices in terms of the variable of integration.
- Write the integral with proper limits.

Step 5: Evaluate the Integral

- Perform the integration carefully.
- Check for convergence; if the integral diverges, the volume does not exist or is infinite.

Complex Cases and Special Considerations

Real-world problems often involve more complicated regions or axes, requiring adjustments:

1. Revolving Around Non-Standard Axes

- Use coordinate transformations or shift the coordinate system to simplify calculations.

2. Regions with Multiple Boundaries

- Break the region into simpler subregions.
- Calculate volumes for each and sum or subtract as needed.

3. Infinite or Unbounded Regions

- Analyze the behavior at infinity.
- Use improper integrals to determine convergence.

Practical Applications and Examples

The concepts of volume of revolution are not just academic; they have widespread applications:

- Engineering: calculating the capacity of tanks, pipes, and mechanical parts.
- Physics: determining moments of inertia or mass distributions.
- Biology: modeling shapes of biological structures.
- Manufacturing: designing objects with rotational symmetry.

Example Problem:

Given the region bounded by $y = \sqrt{x}$ and $y = 0$ between $x=0$ and $x=4$, find the volume of the solid obtained by revolving this region around the x-axis.

Solution Outline:

- Region: $y = \sqrt{x}$, $0 \leq x \leq 4$.
- Axis: x-axis (horizontal).
- Method: Disk method.

Setup:

$$
V = \pi \int_{0}^{4} [\sqrt{x}]^2 \, dx = \pi \int_{0}^{4} x \, dx
$$

Calculation:

$$
V = \pi \left[\frac{x^2}{2} \right]_0^4 = \pi \left(\frac{16}{2} - 0 \right) = 8\pi
$$

The volume is 8π cubic units.

When the Solid of Revolution Does Not Exist

Sometimes, the region or the nature of the functions makes the volume undefined or infinite:

- Unbounded regions extending to infinity typically produce infinite volume unless the functions decay sufficiently fast.
- Discontinuous or non-integrable boundaries prevent setting up a proper integral.
- Revolution around a boundary point that leads to a non-finite integral, such as revolving a region near a vertical asymptote, results in a volume that does not exist in a finite sense.

Example:

Revolving the region between $y=1/x$ and $y=0$ from $x=1$ to $x=\infty$ about the x-axis yields an improper integral. Since:

$$
V = \pi \int_{1}^{\infty} \left(\frac{1}{x} \right)^2 dx = \pi \int_{1}^{\infty} \frac{1}{x^2} dx
$$

which converges to $\pi \left[-\frac{1}{x} \right]_1^\infty = \pi (0 + 1) = \pi$. So, this particular solid has finite volume, but if the region were unbounded in a different manner, the volume could diverge.

Summary

Calculating the volume of a solid of revolution involves a blend of geometric insight and analytical rigor. The key steps include defining the region precisely, choosing the appropriate method (disk/washer or shell), setting up the integral carefully, and evaluating it to determine whether the volume is finite or infinite. The existence of the solid hinges on the boundedness of the region and the behavior of the functions involved.

By understanding these principles, students and practitioners can approach a wide array of problems—from theoretical mathematics to practical engineering—with confidence. Recognizing when a solid of revolution exists and accurately computing its volume can provide valuable insights into the physical structures and systems modeled by these mathematical constructs.

In conclusion, the process of finding the volume of a solid of revolution is a fundamental skill in calculus, requiring careful analysis of the region, the axis of revolution, and the behavior of the functions

involved. When the conditions are right, the methods outlined yield precise results; when they are not, it's essential to recognize that the

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