Find The Work Done By The ForceF=6xyi+3y2jacting Along The Piecewise-smooth Curve Consisting Of The Line

Introduction

Find The Work Done By The Force F=6xyi + 3y²j acting Along The Piecewise-smooth Curve Consisting Of The Line is a classical problem in vector calculus, involving the calculation of work done by a force field along a specified path. This problem combines the concepts of vector fields, line integrals, and piecewise-smooth curves. Understanding how to approach such problems requires familiarity with the fundamental principles of line integrals, parametrization of curves, and the evaluation of integrals in multiple segments. In this article, we will explore the detailed procedure to compute the work done by the given force field along a piecewise-smooth curve, breaking down each step meticulously for clarity.

Understanding the Problem

Force Field Description

The force field under consideration is given by:

- **F(x, y) = 6xy i + 3 y^2 j**

This vector field has components:

- $F_1(x, y) = 6xy$ in the x-direction
- $F_2(x, y) = 3\ y^2$ in the y-direction

Curve Description

The curve is described as a piecewise-smooth path, typically consisting of straight line segments. To compute the work done, the exact parametrization of each segment is necessary. Usually, such problems specify the points or the segments explicitly, but if not, common assumptions include straight lines between known points like (x_1, y_1) to (x_2, y_2), etc.

For illustration, assume the curve consists of two line segments:

1. Segment 1: from point A (x_1, y_1) to point B (x_2, y_2)
2. Segment 2: from point B (x_2, y_2) to point C (x_3, y_3)

In practice, the actual points should be specified, but the method remains the same regardless of the specific points, as long as the curve is piecewise smooth.

Mathematical Approach to the Problem

Line Integral of a Vector Field

The work done by a force field F along a curve C is given by the line integral:

$$W = \int_C F \cdot dr$$

where:

- $F = P(x, y)\, i + Q(x, y)\, j$
- $dr = dx\, i + dy\, j$

Expanding the dot product:

$$F \cdot dr = P(x, y)\, dx + Q(x, y)\, dy$$

Therefore, the work done along a curve C is the sum of integrals over each segment:

$$W = \sum \int_{C_i} [P(x, y)\, dx + Q(x, y)\, dy]$$

Parametrization of the Curve Segments

To evaluate these integrals, each segment must be parametrized. Suppose a segment from point (x_1, y_1) to (x_2, y_2) is parametrized as:

$$x(t) = x_1 + (x_2 - x_1)t$$
$$y(t) = y_1 + (y_2 - y_1)t$$

$$t \in [0, 1]$$

with derivatives:

$$dx/dt = x_2 - x_1$$
$$dy/dt = y_2 - y_1$$

This allows rewriting the integral over each segment as:

$$\int_0^1 [P(x(t), y(t))\, x'(t) + Q(x(t), y(t))\, y'(t)]\, dt$$

Step-by-Step Solution Method

Step 1: Identify Curve Segments and Points

- Determine the points that define the curve segments.
- Verify the parametrization for each segment, ensuring the direction is consistent with the problem statement.

Step 2: Parametrize Each Segment

- Write the parametrization x(t), y(t) for each segment.
- Calculate dx/dt and dy/dt.

Step 3: Express P(x, y) and Q(x, y) Along the Parametrization

- Substitute x(t) and y(t) into P and Q.

Step 4: Set Up the Integral for Each Segment

- Formulate the integral as an integral with respect to t from 0 to 1.
- Compute the integrand: P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t).

Step 5: Integrate and Sum Over All Segments

- Evaluate each integral analytically if possible, or numerically if necessary.
- Add the results to obtain the total work done.

Worked Example

Suppose the curve consists of two segments:

1. Segment 1: from (0, 0) to (1, 1)
2. Segment 2: from (1, 1) to (2, 0)

Parametrization of Segment 1

- **$x(t) = 0 + (1-0)t = t$**
- **$y(t) = 0 + (1-0)t = t$**
- *$t \in [0, 1]$*

Derivatives:

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dx/dt = 1, dy/dt = 1
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Calculating the Work Along Segment 1

- Substitute into F:

$$P(x, y) = 6xy = 6\, t\, t = 6\, t^2$$
$$Q(x, y) = 3\, y^2 = 3\, t^2$$

- Integral becomes:

$$W_1 = \int_0^1 [6\, t^2\, 1 + 3\, t^2\, 1]\, dt = \int_0^1 (6\, t^2 + 3\, t^2)\, dt = \int_0^1 9\, t^2\, dt$$

- Evaluating:

$$W_1 = 9 \int_0^1 t^2\, dt = 9 [t^3/3]_0^1 = 9 (1/3) = 3$$

Parametrization of Segment 2

- $x(t) = 1 + (2 - 1)t = 1 + t$
- $y(t) = 1 + (0 - 1)t = 1 - t$
- $t \in [0, 1]$

Derivatives:

$$dx/dt = 1, \quad dy/dt = -1$$

Calculating the Work Along Segment 2

- Substitute into P and Q:

$$P(x, y) = 6xy = 6 (1 + t) (1 - t) = 6 (1 - t^2)$$
$$Q(x, y) = 3 y^2 = 3 (1 - t)^2$$

- Integral becomes:

$$W_2 = \int_0^1 [6 (1 - t^2) 1 + 3 (1 - t)^2 (-1)]\, dt$$

$$= \int_0^1 [6 (1 - t^2) - 3 (1 - t)^2]\, dt$$

- Expand $(1 - t)^2 = 1 - 2t + t^2$

$$W_2 = \int_0^1 [6 (1 - t^2) - 3 (1 - 2t + t^2)]\, dt = \int_0^1 [6 - 6 t^2 - 3 +$$

Frequently Asked Questions

What is the expression for the force acting along the curve?

The force is given by $F = 6xy\, i + 3\, y^2\, j$.

How do you determine the work done by a force along a piecewise-smooth curve?

The work done is calculated by evaluating the line integral of the force along the curve: $W = \int_C F \cdot dr$, where C is the curve.

What is the significance of the curve being piecewise-smooth in this context?

A piecewise-smooth curve allows the line integral to be computed as a sum of integrals over smooth segments, making the calculation manageable despite potential corners or discontinuities.

How do you parametrize the curve for calculating the work done?

The curve should be parametrized segment-wise with suitable parameterizations $r(t)$, t in $[a, b]$, to evaluate the line integral of the force along each segment.

What are the steps to compute the work done by F along the given curve?

First, parametrize each segment of the curve, then compute the differential dr, evaluate the dot product $F \cdot dr$, and integrate over each segment, summing the results.

Can the work done be zero for certain curves with this force?

Yes, if the force field is conservative or if the curve is along a path where the line integral cancels out, the work done can be zero.

Is the force field $F = 6xy\, i + 3y^2\, j$ conservative?

To determine if F is conservative, check if its curl is zero. In this case, curl $F \neq 0$, so F is not conservative, and the work depends on the path.

How does the piecewise nature of the curve affect the calculation of work?

It requires splitting the integral into segments corresponding to each smooth piece and calculating the line integral over each before summing, ensuring accurate total work.

What are common methods to evaluate the line integral in this problem?

Common methods include parameterization of each curve segment, substitution to simplify the integral, and using fundamental theorem for line integrals if applicable.

Additional Resources

Understanding the Work Done by the Force $\mathbf{F} = 6xy\, \mathbf{i} + 3 y^2\, \mathbf{j}$ Along a Piecewise-Smooth Curve

Analyzing the work done by a force along a specified curve is a fundamental problem in vector calculus, particularly within the context of line integrals. The force field in question, $\mathbf{F} = 6xy\, \mathbf{i} + 3 y^2\, \mathbf{j}$, presents an intriguing scenario owing to its dependence on both x and y. When this force acts along a piecewise-smooth curve, understanding the work done involves delving into concepts such as parameterization, line integrals, potential functions, and the properties of the force field itself. This comprehensive review aims to dissect all relevant aspects, assumptions, and methods to evaluate the work done, providing clarity and depth suitable for advanced studies in vector calculus or physics.

Fundamentals of Work and Line Integrals in Vector Fields

Work Done by a Force Along a Curve

In physics and vector calculus, the work W done by a force $\mathbf{F}$ along a curve C is given by the line integral:

$$
W = \int_C \mathbf{F} \cdot d\mathbf{r}
$$

where:
- $\mathbf{F}$ is the force vector field,
- $d\mathbf{r}$ is an infinitesimal displacement vector along the curve C.

This integral essentially sums the component of the force in the direction of the infinitesimal displacement over the entire path. The work is path-dependent in general, unless the force field is conservative.

Analyzing the Force Field $\mathbf{F} = 6xy\, \mathbf{i} + 3 y^2\, \mathbf{j}$

Properties of the Force Field

The given force field has the following characteristics:
- Components:
$$
F_x = 6xy, \quad F_y = 3 y^2
$$

- It varies with both x and y, indicating a non-uniform field.
- The dependence on xy and y^2 suggests potential symmetry or specific properties that can be exploited.

Checking if the Field is Conservative

A key step in simplifying the work calculation is to determine whether $\mathbf{F}$ is conservative. A force field $\mathbf{F} = P\, \mathbf{i} + Q\, \mathbf{j}$ in two dimensions is conservative if:

$$
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}
$$

Calculating:

$$
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(6xy) = 6x
$$

$$
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(3 y^2) = 0
$$

Since these are not equal ($6x \neq 0$ in general), $\mathbf{F}$ is not conservative. Therefore, the work depends explicitly on the path taken, and a straightforward potential function approach is invalid unless the path is specified, and the integral is computed directly.

Parameterization of the Curve

Piecewise-Smooth Curves

A piecewise-smooth curve is composed of a finite number of smooth segments joined together, each with well-defined tangent vectors. To evaluate the line integral, each segment must be parameterized individually.

Parameterization Strategies

Suppose the curve C consists of segments $C_1, C_2, \ldots, C_n$. For each segment:
- Choose a suitable parameter t over an interval $[a, b]$.

- Express $x(t)$ and $y(t)$ for each segment.
- Compute the differential $d\mathbf{r} = \left(\frac{dx}{dt}, \frac{dy}{dt} \right) dt$.

For example, if a segment is a straight line from (x_1, y_1) to (x_2, y_2):
$$
x(t) = x_1 + (x_2 - x_1) t, \quad y(t) = y_1 + (y_2 - y_1) t, \quad t \in [0,1]
$$
then:
$$
d\mathbf{r} = \left((x_2 - x_1), (y_2 - y_1) \right) dt
$$

The total work is the sum of integrals over each piece:
$$
W = \sum_{i=1}^n \int_{C_i} \mathbf{F} \cdot d\mathbf{r}
$$

Calculating the Line Integral

Explicit Computation Steps

To evaluate $W = \int_C \mathbf{F} \cdot d\mathbf{r}$, follow these steps:

1. Parameterize each curve segment:
- For each segment C_i, find $x(t)$, $y(t)$.
2. Compute derivatives:
- Find $\frac{dx}{dt}$, $\frac{dy}{dt}$.
3. Express $\mathbf{F}$ along the parameter:
- Substitute $x(t)$, $y(t)$ into $\mathbf{F}$:
$$
P(t) = 6 x(t) y(t), \quad Q(t) = 3 y(t)^2
$$
4. Form the integrand:
- The dot product:
$$
\mathbf{F} \cdot d\mathbf{r} = P(t) \frac{dx}{dt} dt + Q(t) \frac{dy}{dt} dt
$$
5. Integrate over the parameter interval:
$$
W_i = \int_{a_i}^{b_i} \left[P(t) \frac{dx}{dt} + Q(t) \frac{dy}{dt} \right]

dt

6. Sum over all segments:

$$
W = \sum_{i=1}^n W_i
$$

Special Cases and Path Choices

Common Path Types

The specific form of the path C significantly influences the integral. Typical paths include:

- Straight lines between two points.
- Circular arcs or other curves.
- Piecewise combinations of lines and curves.

Example: Straight Line from (x_1, y_1) to (x_2, y_2)

Suppose the path is a straight line from (x_1, y_1) to (x_2, y_2):

- Parameterization:

$$
x(t) = x_1 + (x_2 - x_1)t, \quad y(t) = y_1 + (y_2 - y_1)t, \quad t \in [0, 1]
$$

- Derivatives:

$$
\frac{dx}{dt} = x_2 - x_1, \quad \frac{dy}{dt} = y_2 - y_1
$$

- Substituting into the integral:

$$
W = \int_0^1 \left[6 x(t) y(t) (x_2 - x_1) + 3 y(t)^2 (y_2 - y_1) \right] dt
$$

This integral can be evaluated explicitly by expanding the expressions for $x(t)$ and $y(t)$.

Potential Approach for Simplification: Checking for Linearity or Symmetry

Although the force field is non-conservative, certain paths may allow analytic simplifications:
- For paths where \(y \) remains constant, the integrals reduce to single-variable integrals.
- Symmetries in the path or the field may cancel terms or simplify the integral.

Practical Computation and Numerical Methods

In many cases, especially with complex piecewise paths, explicit integration might be cumbersome. Numerical methods provide practical alternatives:
- Quadrature techniques (Simpson's rule, trapezoidal rule).
- Computational tools such as MATLAB, Mathematica, or Python's SciPy library.

These tools enable approximate evaluations, especially when the path segments are complicated or involve non-trivial parameterizations.

Physical Interpretation and Significance

Understanding the work done by the force \(\mathbf{F} \) along the curve has physical implications:
- Energy transfer: The work corresponds to the energy transferred to or from an object moving along \(C \).
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