Find The Work Done In Moving A Particle Once Around A Circle C In The Xy-plane, If The Circle Has Centre

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Understanding the concept of work done in moving a particle along a path is fundamental in vector calculus and physics. When a particle moves around a circle in the xy-plane, especially with a specified center, the analysis involves concepts such as work, force, and vector fields. This article provides a comprehensive explanation of how to determine the work done in one complete revolution of a particle along a circle C in the xy-plane, with emphasis on the case where the circle has a specified center. We will explore the mathematical formulation, methods of calculation, and relevant theorems to understand this problem thoroughly.

Introduction to Work Done in Moving a Particle Along a Path

In physics and mathematics, work done by a force in moving a particle along a path is quantified by the line integral of the force vector field along that path. If $\mathbf{F}$ is the force field and C is the path (in this case, a circle), the work W done is expressed as:

$$
W = \int_C \mathbf{F} \cdot d\mathbf{r}
$$

where:

- $\mathbf{F}$ is the force vector field (possibly derived from a potential or other physical considerations),
- $d\mathbf{r}$ is the differential displacement vector along the path C,
- $\cdot$ denotes the dot product.

The calculation of work depends heavily on the nature of the force field $\mathbf{F}$, the geometry of the path C, and whether the force field is conservative.

Understanding the Geometry of the Circle C

Equation of a Circle in the XY-plane

A circle C in the xy-plane with center at (h, k) and radius r is described by the equation:

$$
(x - h)^2 + (y - k)^2 = r^2
$$

This geometric representation is crucial for parametrization, which simplifies the calculation of line integrals.

Parametrization of the Circle

To evaluate the line integral, it is often best to parametrize the circle. A standard parametrization for a circle centered at (h, k) with radius r is:

$$
\mathbf{r}(t) = h + r \cos t \, \mathbf{i} + k + r \sin t \, \mathbf{j}
$$

where t varies from 0 to 2π.

The derivatives with respect to t are:

$$
\frac{d\mathbf{r}}{dt} = -r \sin t \, \mathbf{i} + r \cos t \, \mathbf{j}
$$

This parametrization is instrumental in converting the line integral over C into an integral over t.

Calculating the Work Done: General Approach

The approach to calculating the work done involves several steps:

1. Identify the Force Field $\mathbf{F}$: Determine the vector field acting on the particle.

2. Parameterize the Path C: Use the parametrization $\mathbf{r}(t)$, with $t \in [0, 2\pi]$.

3. Express $d\mathbf{r}$: Find the differential displacement vector $d\mathbf{r} = \frac{d\mathbf{r}}{dt} dt$.

4. Set up the Line Integral: Write the work as an integral over t:

$$
W = \int_{0}^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} dt
$$

5. Evaluate the Integral: Perform the integral using calculus techniques, considering whether $\mathbf{F}$ is conservative.

Special Cases and Theorems Relevant to Work Calculation

1. Conservative Force Fields

If the force field $\mathbf{F}$ is conservative, then there exists a potential function ϕ such that:

$$
\mathbf{F} = -\nabla \phi
$$

In this case, the line integral over a closed path, such as a circle, simplifies significantly:

$$
W = \oint_C \mathbf{F} \cdot d\mathbf{r} = 0
$$

because the potential function ϕ remains the same after one complete traversal of the circle, leading to zero net work.

Implication: For conservative fields, the work done in moving around a closed loop is zero.

2. Work-Energy Theorem and Path Independence

The work done depends on whether the force field is conservative and the nature of the path:

- Conservative field: Work is path-independent; only depends on initial and final points.
- Non-conservative field: Work depends on the shape of the path; in a closed loop, it can be non-zero.

3. Application of Green's Theorem

Green's theorem relates line integrals around a simple closed curve C to a double integral over the region D bounded by C:

$$
W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy
$$

where:

$$
\mathbf{F} = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}
$$

This theorem simplifies calculations, especially when $\mathbf{F}$ has specific forms.

Calculating Work Done for Specific Force Fields

Case 1: Constant Force Field

Suppose the force field $\mathbf{F}$ is uniform, for example:

$$
\mathbf{F} = P \mathbf{i} + Q \mathbf{j} = A \mathbf{i} + B \mathbf{j}
$$

where A, B are constants.

Calculation:

$$
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \mathbf{F} \cdot \oint_C d\mathbf{r}
$$

Since the integral of $d\mathbf{r}$ around a closed curve is zero, the work done is:

$$
W = 0
$$

unless the force field has a component that does work in the direction of movement.

Case 2: Radial Force Field (Centripetal Force)

If the force is radial, e.g.,

$$
\mathbf{F} = k \mathbf{r}
$$

where k is a constant and $\mathbf{r}$ is the position vector, then the work over one revolution can be calculated using parametrization.

Note: For such force fields, often the work over a closed circle is zero if the force is conservative (which it is in this case).

Step-by-Step Example: Calculating Work for a Specific Force Field

Suppose the force field is:

$$
\mathbf{F}(x, y) = -y \mathbf{i} + x \mathbf{j}
$$

and the circle C has center at the origin $(0,0)$ and radius r.

Step 1: Parametrize the circle:

$$
\mathbf{r}(t) = r \cos t \, \mathbf{i} + r \sin t \, \mathbf{j}
$$

for $t \in [0, 2\pi]$.

Step 2: Compute $d\mathbf{r}$:

$$
\frac{d\mathbf{r}}{dt} = -r \sin t \, \mathbf{i} + r \cos t \, \mathbf{j}
$$

Step 3: Evaluate $\mathbf{F}(\mathbf{r}(t))$:

$$
\mathbf{F}(\mathbf{r}(t)) = -r \sin t \, \mathbf{i} + r \cos t \, \mathbf{j}
$$

which is identical to $\frac{d\mathbf{r}}{dt}$.

Step 4: Compute the dot product:

$$
\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = (-r \sin t)(-r \sin t) + (r \cos t)(r \cos t) = r^2 \sin^2 t + r^2 \cos^2 t = r^2 (\sin^2 t + \cos^2 t) = r^2
$$

Step 5: Set up the integral for work:

$$
W = \int_0^{2\pi} r^2 dt = r^2 (2\pi) = 2 \pi r^2
$$

Result: The work

Frequently Asked Questions

What is the work done in moving a particle once around a circle in the XY-plane when the force is conservative?

The work done is zero because the conservative force's line integral around a closed path is zero, regardless of the circle's center or radius.

How does the position of the circle's center affect the work done in moving a particle around it in the XY-plane?

If the force is conservative and depends only on position, the work done around a closed path is zero, independent of the circle's center location.

What role does the force field's nature play in calculating work done in a circular path in the XY-plane?

In conservative force fields, the work done over a closed loop is zero; in non-conservative fields, it depends on the path and force characteristics.

Can the work done in moving a particle once around a circle in the XY-plane be non-zero if the circle has a specific center?

Yes, if the force field is non-conservative or has non-zero curl, the work done around the circle can be non-zero, regardless of the circle's center.

How is the work done related to the circle's radius and center in a conservative force field?

In a conservative force field, the work done depends only on the initial and final positions; for a closed loop, it is zero, independent of radius and center.

Additional Resources

Work Done in Moving a Particle Once Around a Circle in the XY-Plane

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When delving into the fascinating realm of classical mechanics and vector calculus, one encounters the intriguing concept of work done by a force during particle motion. A common yet profound problem involves calculating the work done in moving a particle once around a circle in the XY-plane, especially when the circle's center and the nature of the force field are specified. This problem not only tests our understanding of vector fields and line integrals but also provides insights into conservative and non-conservative forces, potential energy, and the fundamental principles governing work and energy in physics.

In this comprehensive review, we'll explore this problem through an expert lens, dissecting core concepts, mathematical tools, and the step-by-step approach needed to evaluate the work done in such a scenario.

Understanding the Core Concepts

Before diving into calculations, it’s essential to establish fundamental ideas that underpin this problem.

Work Done by a Force Field

In physics, the work W done by a force $\mathbf{F}$ in moving a particle along a path C from point A to point B is given by the line integral:

$$
W = \int_C \mathbf{F} \cdot d\mathbf{r}
$$

where:

- $\mathbf{F}$ is the force vector field,
- $d\mathbf{r}$ is the infinitesimal displacement vector along the path C.

The physical interpretation of this integral depends on the nature of the force:

- Conservative forces (like gravity or electrostatic forces) have potential functions, and the work done depends only on the endpoints.
- Non-conservative forces (like friction) may depend on the path taken, and the work integral can be path-dependent.

Paths and Circles in the XY-plane

The problem specifies a particle moving once around a circle C in the XY-plane. Key features include:

- Circle C: characterized by its center (x_0, y_0) and radius r.
- The path is closed, meaning the starting and ending point are the same.
- The direction of traversal (clockwise or counterclockwise) can influence the calculation, especially in non-conservative fields.

Nature of the Force Field

Typically, the force field $\mathbf{F}$ in such problems is provided explicitly or described generally. It may be:

- Conservative, e.g., derived from a potential function $V(x, y)$,
- Non-conservative, e.g., involving rotational or curl components.

The nature of $\mathbf{F}$ determines whether the work over a closed path is zero or not.

Mathematical Tools and Theoretical Framework

To evaluate the work done in moving around the circle, several mathematical principles and tools are essential.

Line Integrals and Parametrization

The core calculation involves parametrizing the circle:

$$
x = x_0 + r \cos t, \quad y = y_0 + r \sin t, \quad t \in [0, 2\pi]
$$

and then computing:

$$
W = \int_{t=0}^{2\pi} \mathbf{F}(x(t), y(t)) \cdot \frac{d\mathbf{r}}{dt} dt
$$

where:

$$
d\mathbf{r} = \left(-r \sin t, r \cos t \right) dt
$$

Conservative vs. Non-Conservative Fields

A pivotal aspect is recognizing whether $\mathbf{F}$ is conservative:

- For conservative fields, the work done over any closed path is zero, thanks to the fundamental theorem for line integrals.
- For non-conservative fields, the work may be non-zero, indicating that the force does net work over a cycle.

Mathematically, the curl of $\mathbf{F}$:

\[

$$
\nabla \times \mathbf{F} = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \hat{k}
$$

determines whether $\mathbf{F}$ is conservative:

- If $\nabla \times \mathbf{F} = 0$, the force is conservative.

Use of Green's Theorem

Green's theorem relates the line integral around a simple, closed curve C to a double integral over the region D it encloses:

$$
W = \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx dy
$$

This theorem simplifies calculations, especially when $\nabla \times \mathbf{F}$ is known, and the region D is a circle.

Step-by-Step Approach to Computing the Work Done

Let’s outline the systematic approach to evaluating the work.

Step 1: Define the Force Field $\mathbf{F}$

- Specify $\mathbf{F}(x, y)$. For example, consider:

$$
\mathbf{F}(x, y) = P(x, y) \hat{i} + Q(x, y) \hat{j}
$$

- Clarify whether $\mathbf{F}$ is conservative or not.

Step 2: Parametrize the Circle C

- For a circle with center (x_0, y_0) and radius r:

$$
x(t) = x_0 + r \cos t, \quad y(t) = y_0 + r \sin t, \quad t \in [0, 2\pi]
$$

- The derivatives:

$$
\frac{dx}{dt} = -r \sin t, \quad \frac{dy}{dt} = r \cos t
$$

Step 3: Compute $\mathbf{F}(x(t), y(t))$

- Plug the parametrizations into $\mathbf{F}$:

$$
\mathbf{F}(x(t), y(t)) = P(x(t), y(t)) \hat{i} + Q(x(t), y(t)) \hat{j}
$$

Step 4: Set up the Line Integral

- The work:

$$
W = \int_0^{2\pi} \left[P(x(t), y(t)) \frac{dx}{dt} + Q(x(t), y(t)) \frac{dy}{dt} \right] dt
$$

- Explicitly:

$$
W = \int_0^{2\pi} \left[P(x_0 + r \cos t, y_0 + r \sin t)(-r \sin t) + Q(x_0 + r \cos t, y_0 + r \sin t)(r \cos t) \right] dt
$$

Step 5: Evaluate the Integral

- Use trigonometric identities to simplify integrand.
- Recognize that many terms will integrate to zero over the full cycle, especially if they are sine or cosine functions without matching coefficients.

Step 6: Apply Green's Theorem (Optional for Simplification)

- If $\nabla \times \mathbf{F}$ is constant or known, compute the double integral over the disk D bounded by C:

$$
W = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy
$$

- For a circle of radius r, the area:

$$
A = \pi r^2
$$

- The double integral reduces to:

$$
W = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \times \pi r^2
$$

(assuming this quantity is constant over D).

Case Studies and Practical Examples

To illustrate, consider different types of force fields and their implications.

Example 1: Conservative Force Field

Suppose $\mathbf{F} = \nabla V$, where:

$$
V(x, y) = k \sqrt{(x - x_0)^2 + (y - y_0)^2}
$$

- The force:

$$
\mathbf{F} = -k \frac{(x - x_0, y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}}
$$

- Since $\mathbf{F}$ is conservative, the work over one full circle is zero:

$$
W = 0
$$

because the potential energy depends only on the

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