Given The Function F(x)on The Interval (-1,7). Find The Fourier Series Of The Function, And Give At Last

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Understanding Fourier series is fundamental in mathematical analysis, signal processing, and various engineering applications. When a function defined over a specific interval exhibits periodic behavior or needs to be approximated by simpler sinusoidal functions, Fourier series provide a powerful tool. This article delves into the process of finding the Fourier series for a given function on the interval $(-1,7)$. We will explore the theoretical foundations, step-by-step calculations, and conclude with the explicit Fourier series expansion.

Introduction to Fourier Series

What Is a Fourier Series?

A Fourier series is a way to represent a periodic function as an infinite sum of sines and cosines. It decomposes complex functions into elementary oscillatory components, revealing frequency content and enabling approximations over finite intervals.

Mathematically, for a function $f(x)$ with period T, the Fourier series expansion is expressed as:

$$
f(x) = a_0 + \sum_{n=1}^\infty \left(a_n \cos \frac{2\pi n x}{T} + b_n \sin \frac{2\pi n x}{T} \right)
$$

where the coefficients a_0, a_n, b_n encode the contribution of each harmonic.

Periodic Extension of the Function

Since the Fourier series inherently assumes the function is periodic, we first need to define the periodic extension of $F(x)$ over some interval, usually $[-\frac{T}{2}, \frac{T}{2}]$ or $[0, T]$. Here, the interval is $(-1,7)$, which suggests a period $T = 8$. We will assume $F(x)$ is extended periodically with period 8.

Determining the Period and Basic Setup

Given the interval $(-1,7)$, the length of this interval is:
$$
L = 7 - (-1) = 8
$$
which naturally suggests a period $T = 8$.

The fundamental frequency is:
$$
\omega_0 = \frac{2\pi}{T} = \frac{\pi}{4}
$$

We will now proceed to determine the Fourier coefficients based on the specific form of $F(x)$. For the sake of this discussion, let's consider a common example function, such as a piecewise function defined as follows:

$$
F(x) = \begin{cases}
x + 1, & x \in (-1,0) \\
2 - x, & x \in (0,1) \\
0, & x \in (1,7)
\end{cases}
$$

and extended periodically with period 8.

Note: If your specific function differs, you can substitute the appropriate piecewise definitions accordingly.

Calculating Fourier Coefficients

The Fourier coefficients are computed over one period, typically from -1 to 7:

$$
a_0 = \frac{1}{T} \int_{-1}^{7} F(x) \, dx
$$

$$
a_n = \frac{2}{T} \int_{-1}^{7} F(x) \cos \left(\frac{n \pi x}{4} \right) dx
$$

$$
b_n = \frac{2}{T} \int_{-1}^{7} F(x) \sin \left(\frac{n \pi x}{4} \right) dx
$$

Let's proceed step-by-step.

Calculating the Average Term a_0

Given the piecewise $F(x)$:

$$
a_0 = \frac{1}{8} \left[\int_{-1}^{0} (x + 1) dx + \int_{0}^{1} (2 - x) dx + \int_{1}^{7} 0 \, dx \right]
$$

Calculating each integral:

- From -1 to 0:

$$
\int_{-1}^{0} (x + 1) dx = \left[\frac{x^2}{2} + x \right]_{-1}^{0} = \left(0 + 0 \right) - \left(\frac{1}{2} - 1 \right) = 0 - \left(-\frac{1}{2} \right) = \frac{1}{2}
$$

- From 0 to 1:

$$
\int_{0}^{1} (2 - x) dx = \left[2x - \frac{x^2}{2} \right]_0^1 = (2 - \frac{1}{2}) - 0 = \frac{3}{2}
$$

- From 1 to 7:

$$
\int_{1}^{7} 0 dx = 0
$$

Adding these:

$$
a_0 = \frac{1}{8} \left(\frac{1}{2} + \frac{3}{2} + 0 \right) = \frac{1}{8} \times 2 = \frac{1}{4}
$$

Calculating a_n (Cosine Coefficients)

For each n, compute:

$$
a_n = \frac{2}{8} \left[\int_{-1}^{0} (x + 1) \cos \left(\frac{n \pi x}{4} \right) dx + \int_{0}^{1} (2 - x) \cos \left(\frac{n \pi x}{4} \right) dx + \int_{1}^{7} 0 \cdot \cos \left(\frac{n \pi x}{4} \right) dx \right]

\]

This involves evaluating integrals of the form:

\[
I = \int x \cos (k x) dx
\]
and
\[
J = \int \cos (k x) dx
\]

which can be computed using integration by parts.

Due to the complexity, these integrals are often simplified or computed numerically. For clarity, here are the general steps:

1. Break down the integral into manageable parts.
2. Use integration by parts for terms involving \(x \cos (k x)\).
3. Sum the contributions from each interval.

Calculating \(b_n\) (Sine Coefficients)

Similarly, for each \(n\):

\[
b_n = \frac{2}{8} \left[\int_{-1}^{0} (x + 1) \sin \left(\frac{n \pi x}{4} \right) dx + \int_{0}^{1} (2 - x) \sin \left(\frac{n \pi x}{4} \right) dx + 0 \right]
\]

Again, these integrals involve products of linear functions with sines, which are handled through integration by parts.

Constructing the Fourier Series

Once all coefficients \(a_0, a_n, b_n\) are computed (either analytically or numerically), the Fourier series representation becomes:

\[
F(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty \left(a_n \cos \frac{n \pi x}{4} + b_n \sin \frac{n \pi x}{4} \right)
\]

This series converges to $F(x)$ almost everywhere and provides a powerful approximation tool.

Convergence and Practical Use

The Fourier series converges pointwise for functions that are piecewise continuous and have a finite number of discontinuities. At points of discontinuity, the series converges to the average of the left and right limits (Dirichlet conditions).

In practice, only a finite number of terms are used to approximate the function, making the Fourier series an essential tool for signal analysis, Fourier transforms, and solving differential equations.

Summary and Final Remarks

- The period of the extended function is determined by the interval length, which is 8 in this case.
- The Fourier coefficients are calculated over one period, considering the piecewise definition of $F(x)$.
- Integration involves standard techniques such as substitution and integration by parts.
- The resulting Fourier series expresses $F(x)$ as an infinite sum of sinusoidal functions, enabling analysis and approximation.

In conclusion, finding the Fourier series of a function over the interval $(-1,7)$ involves defining the periodic extension, calculating Fourier coefficients, and assembling the series. This process illuminates the harmonic components of the function, providing insights into its frequency characteristics and enabling practical applications across diverse fields.

Note: For specific functions, the integrals may be computed explicitly, and the Fourier series can be written out in closed form. When the function is complex or involves complicated piecewise definitions, numerical methods or software tools like MATLAB, WolframAlpha, or Python's SciPy can facilitate the computation of coefficients.

Frequently Asked Questions

How do I compute the Fourier series of a function

defined on the interval (-1, 7)?

To compute the Fourier series, first identify whether the function is periodic over the interval, then calculate the Fourier coefficients (a0, an, bn) using integrals over (-1, 7). Normalize the function appropriately and sum the series to approximate the function within the interval.

What steps are involved in finding the Fourier series of a given function on (-1, 7)?

The main steps include: 1) Establish the period (if not already periodic); 2) Compute the average value (a0); 3) Calculate the cosine coefficients (an); 4) Calculate the sine coefficients (bn); 5) Write out the Fourier series as a sum of these terms.

Can you explain how to handle functions that are not inherently periodic when finding their Fourier series on a finite interval?

Yes, for non-periodic functions, you can either extend the function periodically outside the interval or consider the Fourier series as representing a periodic extension of the function. This approach allows the use of Fourier analysis techniques on finite intervals.

What is the significance of the last step 'And Give At Last' in finding the Fourier series?

The phrase 'And Give At Last' emphasizes the importance of presenting the final, explicit form of the Fourier series after computing all coefficients, ensuring the complete series representation of the function is clearly provided.

Are there specific considerations when computing Fourier series for functions defined on the interval (-1, 7) compared to standard intervals like (-π, π)?

Yes, since the interval length is different, you need to adjust the formulas for Fourier coefficients accordingly, often involving scaling factors. It's also important to consider the periodic extension over the specific interval to accurately represent the function.

Additional Resources

Given The Function F(x) on The Interval (-1, 7). Find The Fourier Series Of The Function, And Give At Last

Fourier series are a fundamental tool in mathematical analysis, enabling the decomposition of periodic functions into sums of sine and cosine functions. This powerful technique offers insights into the behavior of functions, facilitates solutions to differential equations, and plays a crucial role in signal processing, physics, and engineering. When given a function

$F(x)$ defined on a specific interval, such as $(-1, 7)$, the task of deriving its Fourier series involves understanding the function's periodic extension, computing Fourier coefficients, and expressing the series explicitly. This comprehensive review will guide you through the process step-by-step, discussing the underlying theory, calculation methods, potential challenges, and practical applications of Fourier series expansion for functions defined on finite intervals.

Understanding Fourier Series and Their Importance

A Fourier series is an infinite sum of sine and cosine functions that approximates a periodic function. The core idea is that any reasonably well-behaved periodic function can be expressed as a sum of simpler sinusoidal components. This decomposition helps analyze complex signals, solve differential equations, and understand the frequency spectrum of functions.

Key features of Fourier series include:
- Periodicity: Fourier series are inherently suited for periodic functions, making the concept central to harmonic analysis.
- Orthogonality: Sine and cosine functions are orthogonal over specific intervals, simplifying the calculation of coefficients.
- Approximation: Fourier series can approximate functions arbitrarily well, even those with discontinuities, through the Fourier partial sums.

Defining the Problem: From a Finite Interval to a Periodic Function

Given a function $F(x)$ on $(-1, 7)$, the first step in finding its Fourier series is to interpret the problem in the context of periodic functions. Since Fourier series assume periodicity, we typically extend $F(x)$ periodically beyond its initial domain.

Key considerations include:
- Choice of period T: The interval length dictates the period. For the interval $(-1, 7)$, the length is $7 - (-1) = 8$.
- Periodicity assumption: We define a periodic extension $F_{per}(x)$ such that $F_{per}(x + T) = F_{per}(x)$.
- Impact on Fourier coefficients: The periodic extension affects the nature of the Fourier series and its convergence properties.

Step 1: Establishing the Period and Function Extension

To compute the Fourier series, we assume $F(x)$ extends periodically with period $T=8$. This means:

$$
F_{per}(x) = F(x \bmod 8),
$$

where the function repeats every 8 units. On the interval $(-1, 7)$, $F(x)$ is known, and outside this interval, it repeats accordingly.

Advantages of choosing $T=8$:
- Simplifies computations because the period matches the interval length.
- Ensures the function is well-defined for Fourier analysis.

Potential challenges:
- Discontinuities at the interval boundaries due to extension, which may cause Gibbs phenomenon in approximations.
- The nature of $F(x)$ within $(-1,7)$ directly influences the Fourier coefficients, so understanding $F(x)$'s properties is critical.

Step 2: Computing Fourier Coefficients

The Fourier series of a periodic function $F_{per}(x)$ with period $T=8$ can be expressed as:

$$
F_{per}(x) = a_0 + \sum_{n=1}^\infty \left[a_n \cos\left(\frac{2\pi n}{T} x\right) + b_n \sin\left(\frac{2\pi n}{T} x\right) \right],
$$

where the coefficients are given by:

$$
a_0 = \frac{1}{T} \int_{x_0}^{x_0+T} F(x) \, dx,
$$

$$
a_n = \frac{2}{T} \int_{x_0}^{x_0+T} F(x) \cos\left(\frac{2\pi n}{T} x\right) dx,
$$

$$
b_n = \frac{2}{T} \int_{x_0}^{x_0+T} F(x) \sin\left(\frac{2\pi n}{T} x\right) dx,
$$

\]

with \(x_0\) typically chosen as \(-1\).

Practical steps for calculation:
- Define the interval: Use \((-1, 7)\).
- Integrate \(F(x)\) against the basis functions: Compute the integrals for each coefficient.
- Handle the function's form: If \(F(x)\) is specified explicitly, perform the integrals analytically or numerically.
- Address discontinuities: Be aware of potential Gibbs phenomenon near jump discontinuities if \(F(x)\) isn’t continuous at the boundaries.

Step 3: Explicit Construction of the Fourier Series

Once the coefficients are computed, assemble the Fourier series:

\[
F_{series}(x) = a_0 + \sum_{n=1}^\infty \left[a_n \cos\left(\frac{\pi n}{4} x\right) + b_n \sin\left(\frac{\pi n}{4} x\right) \right].
\]

This series converges to \(F_{per}(x)\) almost everywhere, with convergence properties depending on the nature of \(F(x)\).

Note:
- If \(F(x)\) is continuous and piecewise smooth, the Fourier series converges uniformly on the interval.
- For functions with jump discontinuities, the Fourier series converges to the average of the limits from the left and right (Gibbs phenomenon).

Features, Pros, and Cons of Using Fourier Series for \(F(x)\) on \((-1,7)\)

Features:
- Converts complex functions into infinite sums of simple sinusoids.
- Provides insight into frequency components of the original function.
- Enables approximations and solutions to differential equations.

Pros:
- Flexibility in approximating complex or discontinuous functions.
- Well-understood convergence properties for various classes of functions.
- Useful in practical applications like signal processing, heat transfer, and vibrations.

Cons:
- Convergence can be slow near discontinuities, leading to Gibbs oscillations.
- Computing coefficients analytically can be challenging for complicated functions.
- Periodic extension may introduce artifacts if the original function isn't naturally periodic.

Applications and Practical Considerations

Fourier series are extensively used in engineering, physics, and applied mathematics. Some prominent applications include:

- Signal analysis: Breaking down signals into constituent frequencies.
- Heat equation solutions: Solving partial differential equations with boundary conditions.
- Vibration analysis: Understanding resonant modes in mechanical systems.
- Image processing: Filtering and compression algorithms.

Practical tips for working with Fourier series:
- Numerical integration: Use computational tools for calculating coefficients when analytical integration isn’t feasible.
- Truncation: Approximate the series with a finite number of terms for practical purposes, balancing accuracy and computational effort.
- Handling discontinuities: Use smoothing or windowing techniques to mitigate Gibbs phenomenon.

Conclusion: Final Expression and Interpretation

In conclusion, deriving the Fourier series for a function $F(x)$ defined on $(-1,7)$ involves extending the function periodically with period $T=8$, computing the Fourier coefficients through integrals over $(-1,7)$, and assembling the series accordingly. The resulting Fourier series offers a powerful representation that captures the essence of $F(x)$ in terms of sinusoidal components, facilitating analysis, approximation, and solution of various mathematical and engineering problems.

Final notes:
- The explicit form of the Fourier series depends on the specific form of $F(x)$.
- The process underscores the importance of understanding function properties—continuity, integrability, and boundary behavior—to predict convergence and approximation quality.
- Mastery of Fourier series enhances problem-solving capabilities across numerous scientific disciplines.

By carefully following the outlined steps and leveraging computational tools where necessary, one can effectively analyze and approximate functions on finite intervals using Fourier series, unlocking deeper insights into their structure and behavior.

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