Given The System Of Inequalities Below, Determine The Shape Of The Feasible Region And Find The Vertices

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Understanding how to analyze systems of inequalities is a fundamental skill in linear programming and optimization. Such systems help visualize feasible solutions within certain constraints, enabling decision-makers, students, and professionals to identify optimal solutions efficiently. This article provides a comprehensive guide to determine the shape of the feasible region and find its vertices based on a given system of inequalities.

Understanding the Basics of Systems of Inequalities

Before diving into the process, it is essential to understand what a system of inequalities entails and how it defines a feasible region.

What Are Inequalities?

An inequality is a mathematical statement that compares two expressions using symbols such as <, >, □, or □. For example:

- 2x + y □ 10
- x □ 0
- y > 3

Each inequality restricts the possible values of variables x and y in some way.

What Is a System of Inequalities?

A system of inequalities combines multiple inequalities, and the feasible region is the set of all points (x, y) that satisfy all inequalities simultaneously.

Example:

$$
\begin{cases}
x + y \leq 8 \\
x - y \geq 2 \\
x \geq 0 \\
y \geq 0
\end{cases}
$$

The goal is to analyze this system: determine the shape formed by the intersection of the regions defined by each inequality and identify the corner points (vertices) of this shape.

Step-by-Step Process to Determine the Feasible Region and Find Vertices

The process involves graphing the inequalities, identifying the feasible region, and then calculating the vertices, which are critical in optimization problems.

Step 1: Rewrite Inequalities in Slope-Intercept Form

Transform each inequality into the form $y = mx + b$ if possible, as this facilitates graphing.

- For example, $x + y \leq 8$ becomes $y \leq -x + 8$.
- $x - y \geq 2$ becomes $y \leq x - 2$ (note the inequality direction change).

Step 2: Graph Each Boundary Line

- Draw the boundary lines corresponding to the equations (by replacing inequality signs with =).
- Use solid lines for $\leq$ and $\geq$ to indicate the boundary is included; dashed lines for $<$ and $>$ if the boundary is not included.
- Mark the intercepts and slope for each line for accurate graphing.

For the example:
- Line 1: $x + y = 8$ (solid line), with intercepts at (8, 0) and (0, 8).
- Line 2: $x - y = 2$ (solid line), with intercepts at (2, 0) and (0, -2).

Step 3: Shade the Regions Corresponding to Each Inequality

- For each inequality, shade the region that satisfies it.
- For $y \leq -x + 8$, shade below the line.
- For $y \leq x - 2$, shade below the line.

Additionally, for the inequalities $x \geq 0$ and $y \geq 0$, shade the regions to the right of the y-axis and above the x-axis respectively.

Step 4: Identify the Feasible Region

- The feasible region is the intersection of all shaded regions.
- Usually, this region is a polygon (triangle, quadrilateral, etc.) that contains all possible solutions

satisfying the system.

Step 5: Find the Vertices of the Feasible Region

Vertices are the intersection points of the boundary lines. To find these:

- Solve the equations of the boundary lines pairwise.
- Check whether the intersection points lie within the shaded regions.
- Include the points where the boundary lines intersect with the axes if they satisfy all inequalities.

Example:

Solve for the intersection of:

- $x + y = 8$
- $x - y = 2$

Adding the two equations:

$$
(x + y) + (x - y) = 8 + 2 \Rightarrow 2x = 10 \Rightarrow x=5
$$

Substitute $x=5$ into $x + y=8$:

$$
5 + y=8 \Rightarrow y=3
$$

Vertex: (5,3)

Similarly, find all other intersections, such as:

- Intersection with axes (e.g., $x=0$, $y=0$)
- Intersection between the boundary lines and axes, checking if they satisfy all inequalities.

Determining the Shape of the Feasible Region

The shape of the feasible region depends on the inequalities involved. Common shapes include:

- Triangle: When three lines intersect to form a bounded triangular area.
- Quadrilateral: When four boundary lines intersect, forming a four-sided polygon.
- Polygon with more sides: If multiple inequalities intersect to create more complex polygons.
- Unbounded region: If the feasible region extends infinitely in one or more directions.

Factors Influencing the Shape

- The number of inequalities and their orientations.
- Whether the inequalities form a closed or open region.
- The presence of strict inequalities (<, >) which exclude boundary points.

Example: Complete Analysis of a System of Inequalities

Suppose the system is:

$$
\begin{cases}
x + 2y \leq 10 \\
3x + y \geq 8 \\
x \geq 0 \\
y \geq 0
\end{cases}
$$

Step 1: Rewrite in slope-intercept form:

- $x + 2y = 10 \Rightarrow y = \frac{10 - x}{2}$
- $3x + y = 8 \Rightarrow y = 8 - 3x$

Step 2: Graph the boundary lines:

- Line 1: $y = -\frac{1}{2}x + 5$, solid line.
- Line 2: $y = -3x + 8$, solid line.

Step 3: Shade the regions:

- For $x + 2y \leq 10$: shade below $y = -\frac{1}{2}x + 5$.
- For $3x + y \geq 8$: shade above $y = -3x + 8$.
- For $x \geq 0$: right of the y-axis.
- For $y \geq 0$: above the x-axis.

Step 4: Find intersection points:

- Intersection of the two lines:

Set $-\frac{1}{2}x + 5 = -3x + 8$:

$$
-\frac{1}{2}x + 5 = -3x + 8
$$

Multiply through by 2:

$$
- x + 10 = -6x + 16
$$

Bring all to one side:

$$
- x + 6x = 16 - 10
$$

$$
5x = 6

$$

$$
x = \frac{6}{5} = 1.2
$$

Calculate y:

$$
y = -3(1.2) + 8 = -3.6 + 8 = 4.4
$$

Vertex: $\left(\frac{6}{5}, 4.4\right)$

- Intersection with axes:

- When $x=0$:
- $y \leq 5$ from first inequality.
- $y \geq \frac{8 - 3(0)}{1} = 8$, but since $y \leq 5$ from the first line, the feasible y at $x=0$ must satisfy both. The maximum y is 5, but the second inequality requires $y \geq 8$. Since 8 > 5, the feasible region does not include the point $(0,8)$. So, no feasible point at $x=0, y \geq 8$.

- When $y=0$:
- $x \leq 10$ from the first inequality.
- $x \geq \frac{8 - y}{3} = \frac{8}{3} \approx 2.67$.
- Also, $x \geq 0$, so feasible points at $y=0$ are between $x \approx 2.67$ and 10.

Step 5: Final feasible region

- The shape is a polygon bounded by:
- The intersection point $\left(\frac{6}{5}, 4.4\right)$,
- The intersection points with axes,
- And the boundary lines.

- The vertices are:

- $\left(\frac{6}{5}, 4.4\right)$,
- The points where the lines intersect the axes satisfying all inequalities,
- And possibly

Frequently Asked Questions

How do you identify the shape of the feasible region when given a system of inequalities?

The shape of the feasible region is determined by graphing each inequality and finding the overlapping area that satisfies all conditions. Typically, the feasible region is a polygon—such as a triangle, quadrilateral, or other polygon—formed by the boundary lines of the inequalities.

What is the significance of the vertices in the feasible region for linear inequalities?

Vertices of the feasible region are the intersection points of the boundary lines. They are critical because, according to the Corner Point Theorem, the maximum or minimum values of a linear objective function occur at one of these vertices.

How can I find the vertices of the feasible region from the given inequalities?

Vertices are found by solving the systems of equations formed by the boundary lines of the inequalities that intersect. Each pair of boundary lines is solved simultaneously to find their intersection point, which becomes a vertex if it satisfies all inequalities.

What role does graphing each inequality play in determining the

feasible region?

Graphing each inequality helps visualize the region that satisfies all constraints. By shading the feasible side of each boundary and identifying the overlapping shaded area, you can accurately determine the shape and extent of the feasible region.

Can the feasible region be unbounded, and how does that affect the vertices?

Yes, the feasible region can be unbounded if the inequalities do not restrict the region in all directions. In such cases, the vertices are finite intersection points, but the region extends infinitely in at least one direction beyond these vertices.

What methods can be used to find the exact coordinates of the vertices?

You can find the vertices by solving the systems of linear equations formed by the boundary lines of the inequalities. Techniques include substitution, elimination, or using matrix methods like Cramer's rule for precise solutions.

Why is understanding the shape and vertices of the feasible region important in solving linear programming problems?

Because the optimal solutions to linear programming problems occur at the vertices of the feasible region, understanding its shape and vertices allows you to evaluate the objective function at these points efficiently and determine the maximum or minimum values.

Additional Resources

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Introduction

In the realm of optimization and linear programming, understanding the feasible region defined by a system of inequalities is fundamental. The feasible region represents all the possible solutions that satisfy a set of constraints, and its shape, boundaries, and vertices are essential for solving problems such as maximizing profit, minimizing cost, or optimizing resource allocation. In this investigative article, we delve into the process of analyzing a given system of inequalities to determine the shape of the feasible region and identify its vertices comprehensively.

The Significance of Feasible Regions in Linear Programming

Before we explore the specifics of the inequalities, it’s vital to understand why the feasible region holds such importance:

- Solution Space Visualization: It provides a visual representation of all potential solutions.
- Vertex Theorem: The optimal solution to many linear programming problems lies at a vertex (corner point) of the feasible region.
- Constraint Analysis: It helps identify binding constraints and redundant inequalities.

The System of Inequalities Under Investigation

Let us consider a generic system of inequalities involving two variables, \(x \) and \(y \):

\[
\begin{cases}

a_1x + b_1y \leq c_1 \\

a_2x + b_2y \leq c_2 \\

a_3x + b_3y \leq c_3 \\

x \geq 0 \\

y \geq 0

\end{cases}

\]

Note: For illustrative purposes, suppose the system is:

\[

\begin{cases}

x + 2y \leq 8 \quad (1) \\

3x + y \leq 9 \quad (2) \\

x \geq 0 \quad (3) \\

y \geq 0 \quad (4)

\end{cases}

\]

This set of inequalities models a typical feasible region scenario in the first quadrant bounded by two lines.

Step 1: Graphical Representation of the Constraints

Plotting the Boundary Lines

Each inequality corresponds to a boundary line:

- Line from (1): \(x + 2y = 8 \)

- Line from (2): $3x + y = 9$

The inequalities specify which side of each boundary line the feasible region occupies:

- $x + 2y \leq 8$ implies the region below or on the line.
- $3x + y \leq 9$ implies the region below or on the line.
- $x \geq 0$ and $y \geq 0$ restrict the feasible region to the first quadrant.

Key Points for Boundary Lines

- For $x + 2y = 8$:
- When $x=0$, $y=4$.
- When $y=0$, $x=8$.

- For $3x + y = 9$:
- When $x=0$, $y=9$.
- When $y=0$, $x=3$.

Plotting these lines on the coordinate plane forms the initial shape of the feasible region.

Step 2: Determining the Shape of the Feasible Region

Analysis of Constraints

The feasible region is the intersection of all half-planes defined by the inequalities:

- The region below $x + 2y=8$,
- The region below $3x + y=9$,
- The region in the first quadrant ($x, y \geq 0$).

Shape Identification

Given the intersection of these regions, the feasible region is a polygon, specifically a convex quadrilateral or triangle depending on the constraints.

- The boundary lines intersect at a point, forming vertices.
- The constraints $x \geq 0$ and $y \geq 0$ limit the region to the first quadrant.
- The intersection points and the boundary lines form a polygon with straight edges.

Expected Shape:

- Since the two lines intersect within the first quadrant, and the axes form the other boundaries, the feasible region is a convex polygon, most likely a quadrilateral.

Step 3: Finding the Vertices of the Feasible Region

Vertices are critical because, by the Fundamental Theorem of Linear Programming, the optimal solutions occur at these points.

Vertices are formed at the intersections of the boundary lines and axes:

1. Origin point: $(0,0)$, where $x=0$ and $y=0$.

2. Intersection with axes:
- From $x + 2y=8$:
- $(0,4)$
- $(8,0)$
- From $3x + y=9$:
- $(0,9)$

- $(3,0)$

3. Intersections of the boundary lines:

Solve the system:

$$
\begin{cases}
x + 2y = 8 \\
3x + y = 9
\end{cases}
$$

Express y from the second:

$$
y = 9 - 3x
$$

Substitute into the first:

$$
x + 2(9 - 3x) = 8 \Rightarrow x + 18 - 6x = 8
$$

Simplify:

$$
-5x = -10 \Rightarrow x=2
$$

Calculate y:

$$
y= 9 - 3(2) = 9 - 6=3
$$

Vertex of intersection: $(2,3)$

Step 4: Confirming the Feasible Vertices

Considering the constraints $x \geq 0, y \geq 0$, and the inequalities, the feasible vertices are:

- $(0,0)$
- $(0,4)$
- $(2,3)$
- $(3,0)$
- $(8,0)$ (though outside the feasible region, as it does not satisfy $3x + y \leq 9$)
- $(0,9)$ (similarly outside the feasible region)

Verification of each point:

- $(0,0)$: Satisfies all.
- $(0,4)$: Check $3(0)+4=4 \leq 9$, OK; $0 + 2(4)=8 \leq 8$, OK.
- $(2,3)$: $2 + 2(3)=2+6=8 \leq 8$, OK; $3(2)+3=6+3=9 \leq 9$, OK.
- $(3,0)$: $3+0=3 \leq 8$, OK; $3(3)+0=9 \leq 9$, OK.

Points like $(8,0)$ and $(0,9)$ violate the constraints and are outside the feasible region.

Step 5: Visual Representation and Confirmation

Plotting these vertices and the boundary lines yields a polygon with vertices at:

- \((0,0) \)
- \((0,4) \)
- \((2,3) \)
- \((3,0) \)

This convex polygon is the feasible region, specifically a quadrilateral bounded by:

- The axes,
- The line \(x + 2y = 8 \),
- The line \(3x + y = 9 \).

Conclusion: The Shape and Vertices of the Feasible Region

The systematic analysis of the system of inequalities reveals that the feasible region is a convex quadrilateral situated in the first quadrant, bounded by the axes and the two lines:

- Shape: Convex quadrilateral (parallelogram or irregular quadrilateral).
- Vertices:
- \((0,0) \)
- \((0,4) \)
- \((2,3) \)
- \((3,0) \)

These vertices are critical in optimization tasks, as they represent the potential points where the objective function may attain its maximum or minimum.

Implications and Practical Applications

Understanding the shape and vertices of the feasible region allows practitioners to:

- Efficiently evaluate the objective function at each vertex to identify optimal solutions.
- Visualize constraints in multi-dimensional problems.
- Detect redundant or non-binding constraints for model simplification.
- Extend analysis to more complex systems involving additional variables or constraints.

Final Thoughts

This investigative approach underscores the importance of geometric reasoning in linear programming. By systematically plotting inequalities, determining the intersection points, and confirming the shape of the feasible region, analysts and decision-makers can confidently identify critical solution points, streamline problem-solving processes, and derive optimal strategies in practical scenarios. Whether in manufacturing, logistics, or finance, mastering the analysis of feasible regions is indispensable for effective decision-making.

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