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If Z = (x + y) e^y, X = 3t, Y = 3t^2, Find Dz/dt Using The Chain Rule. Assume The Variables Are Restricted

In the realm of calculus, understanding how to differentiate composite functions is fundamental. When dealing with functions of multiple variables that themselves depend on other variables, the chain rule becomes an essential tool. This article delves into a specific problem involving the function $Z = (x + y) e^y$, with x and y expressed in terms of a parameter t. We aim to find $\frac{dZ}{dt}$ using the chain rule, considering the given relationships $X = 3t$ and $Y = 3t^2$.

This problem not only reinforces the application of the multivariable chain rule but also emphasizes the importance of understanding variable dependencies, especially when variables are restricted or parametrized. By the end of this article, you will have a clear step-by-step approach to compute $\frac{dZ}{dt}$, along with insights into the underlying calculus principles.

Understanding the Problem Context

Before diving into the solution, it’s crucial to understand the problem's components:

- The function Z depends on two variables, x and y:

$$
Z = (x + y) e^y
$$

- The variables x and y are functions of a parameter t:

$$
x = 3t, \quad y = 3t^2
$$

- The goal is to find the derivative of Z with respect to t, denoted as $\frac{dZ}{dt}$.

This scenario exemplifies a typical multivariable calculus problem where the chain rule facilitates differentiation of composite functions involving multiple layers of dependency.

Fundamentals of the Chain Rule in Multivariable Calculus

The chain rule extends beyond single-variable functions, providing a systematic way to differentiate composite functions involving multiple variables. When a function Z depends on variables x and y, which themselves depend on t, the total derivative $\frac{dZ}{dt}$ can be expressed as:

$$
\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}
$$

Here:

- $\frac{\partial Z}{\partial x}$ and $\frac{\partial Z}{\partial y}$ are the partial derivatives of Z with respect to x and y, respectively.
- $\frac{dx}{dt}$ and $\frac{dy}{dt}$ are the derivatives of x and y with respect to t.

This formula captures how changes in t influence Z through the pathways of $x(t)$ and $y(t)$.

Step-by-Step Solution to Find $\frac{dZ}{dt}$

Let's now proceed with the detailed steps to compute $\frac{dZ}{dt}$:

Step 1: Write down the known functions

- $Z = (x + y) e^y$
- $x(t) = 3t$
- $y(t) = 3t^2$

Step 2: Compute the partial derivatives of Z with respect to x and y

Since $Z = (x + y) e^y$, treat x and y as independent variables:

- Partial derivative with respect to x:

\[

$$
\frac{\partial Z}{\partial x} = e^y
$$

- Partial derivative with respect to y:

$$
\frac{\partial Z}{\partial y} = \frac{\partial}{\partial y} \left((x + y) e^y \right)
$$

Apply the product rule:

$$
\frac{\partial Z}{\partial y} = (x + y) \frac{\partial}{\partial y} e^y + e^y \frac{\partial}{\partial y} (x + y)
$$

$$
= (x + y) e^y + e^y (0 + 1) = (x + y) e^y + e^y
$$

$$
= e^y (x + y + 1)
$$

Step 3: Compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$

- $x(t) = 3t \Rightarrow \frac{dx}{dt} = 3$
- $y(t) = 3t^2 \Rightarrow \frac{dy}{dt} = 6t$

Step 4: Substitute all derivatives into the chain rule formula

$$
\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}
$$

Expressed explicitly:

$$
\frac{dZ}{dt} = e^y \times 3 + e^y (x + y + 1) \times 6t
$$

Now, substitute $x(t)$ and $y(t)$:

$$
x = 3t, \quad y = 3t^2
$$

So:

$$
\frac{dZ}{dt} = e^{3t^2} \times 3 + e^{3t^2} \times (3t + 3t^2 + 1) \times 6t
$$

Factor out e^{3t^2}:

$$
\frac{dZ}{dt} = e^{3t^2} \left[3 + (3t + 3t^2 + 1) \times 6t \right]
$$

Final Expression for $\frac{dZ}{dt}$

The explicit formula for the derivative $\frac{dZ}{dt}$, incorporating all components, is:

$$
\boxed{
\frac{dZ}{dt} = e^{3t^2} \left[3 + 6t (3t + 3t^2 + 1) \right]
}
$$

This expression can be expanded further if necessary, but often leaving it in factored form aids in analysis and understanding.

Analyzing the Results and Special Cases

Understanding the behavior of $\frac{dZ}{dt}$ at specific points can offer insights:

- When $t = 0$:

$$
\frac{dZ}{dt} \bigg|_{t=0} = e^{0} \times [3 + 0 \times (0 + 0 + 1)] = 1 \times 3 = 3
$$

- For large t:

The exponential term e^{3t^2} dominates, causing the derivative to grow rapidly, indicating rapid change in Z as t increases.

Assumptions and Variable Restrictions

In solving this problem, certain assumptions about the variables and their relationships are made:

- The functions $x(t)$ and $y(t)$ are differentiable over the domain of interest.
- The exponential function e^y is well-defined for all real y.
- The variables are continuous and differentiable, allowing the application of the chain rule.
- The problem assumes no restrictions on t other than those ensuring the functions are defined and differentiable.

These assumptions are standard in calculus problems involving continuous and differentiable functions.

Conclusion and Summary

This comprehensive exploration demonstrates the application of the chain rule to find $\frac{dZ}{dt}$ for the function $Z = (x + y) e^y$, with $x = 3t$ and $y = 3t^2$. The key steps involved computing the partial derivatives of Z, differentiating the parametrized variables, and applying the chain rule formula. The final expression highlights how the rate of change of Z depends on both the exponential growth and the polynomial components.

Understanding such derivative computations is vital in fields like physics, engineering, and economics, where multivariable dependencies are common. Mastery of the chain rule enables analysts to navigate complex models involving interconnected variables effectively.

In practice, this approach can be extended to more intricate functions and higher-dimensional systems, reinforcing the importance of foundational calculus concepts in advanced applications.

Keywords: Chain Rule, Multivariable Calculus, Partial Derivatives, Derivative of Composite Functions, Parametric Functions, Derivative with Respect to t, Calculus Problems, Exponential Functions, Variable Dependencies

Frequently Asked Questions

What is the expression for Z in terms of x and y?

Z = (x + y) e^y

Given x = 3t and y = 3t^2, how do we find dZ/dt using the

chain rule?

We differentiate Z with respect to t by applying the chain rule: $dZ/dt = (\partial Z/\partial x)(dx/dt) + (\partial Z/\partial y)(dy/dt)$.

What are the partial derivatives $\partial Z/\partial x$ and $\partial Z/\partial y$ for $Z = (x + y) e^y$?

$\partial Z/\partial x = e^y$ and $\partial Z/\partial y = (x + y) e^y + e^y = (x + y + 1) e^y$.

What are dx/dt and dy/dt for the given functions?

$dx/dt = 3$ and $dy/dt = 6t$.

How do you compute dZ/dt at a specific value of t?

Substitute $x = 3t$ and $y = 3t^2$ into the partial derivatives and multiply by dx/dt and dy/dt respectively, then sum the results.

Is there any restriction on the variables when applying the chain rule here?

Yes, the problem states variables are restricted, implying differentiability and that the functions are well-defined within the domain considered.

What is the step-by-step process to find dZ/dt for the given functions?

First, find $\partial Z/\partial x$ and $\partial Z/\partial y$; second, compute dx/dt and dy/dt; third, apply the chain rule: $dZ/dt = (\partial Z/\partial x)(dx/dt) + (\partial Z/\partial y)(dy/dt)$; finally, substitute x and y in terms of t and simplify.

Can you provide the final expression for dZ/dt in terms of t?

Yes. Substituting $x = 3t$ and $y = 3t^2$, we get $dZ/dt = e^{3t^2}\, 3 + [(3t + 3t^2 + 1) e^{3t^2}]\, 6t$.

Why is the chain rule essential in solving this problem?

Because Z depends on x and y, which in turn depend on t, the chain rule allows us to account for these dependencies to find the rate of change of Z with respect to t.

Additional Resources

Understanding the Chain Rule in Multivariable Calculus: A Deep Dive into Derivatives with Variable Dependence

In the realm of calculus, particularly when dealing with functions of multiple variables, the chain rule serves as an essential tool for computing derivatives of composite functions. When the variables themselves are functions of an independent parameter, such as time, the process becomes more nuanced and requires a systematic approach to track how changes propagate through the interconnected variables. This article explores a specific scenario involving the function $Z = (x + y) e^y$, with the variables x and y expressed as functions of a parameter t, namely $X = 3t$ and $Y = 3t^2$. We aim to find the derivative of Z with respect to t, denoted as Dz/dt, using the chain rule, considering the restrictions and dependencies among the variables.

Fundamental Concepts: Multivariable Functions and the Chain Rule

Multivariable Functions and Their Dependencies

In multivariable calculus, functions often depend on several independent variables. For example, Z = Z(x, y) indicates that Z is a function of x and y, which may themselves be functions of other variables or parameters. When x and y depend on a common parameter t (such as time), the overall change in Z with respect to t reflects how both x and y evolve over t, influencing Z simultaneously.

Mathematically, this is expressed as:

$$ Z = Z(x(t), y(t)) $$

The derivatives of Z with respect to t must account for the indirect dependence of Z on t through x and y.

The Chain Rule in Multiple Variables

The chain rule extends to functions of multiple variables through the following formula:

$$
\frac{dZ}{dt} = \frac{\partial Z}{\partial x} \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}
$$

This expression captures the total rate of change of Z with respect to t by summing the contributions from x and y, weighted by how sensitive Z is to each variable ($\partial Z / \partial x$ and $\partial Z / \partial y$), and how quickly each variable changes with t ($dx/dt, dy/dt$).

Given Functions and Their Interrelations

Defined Variables and Their Dependence on t

In this scenario, the functions are specified as:

- $X(t) = 3t$
- $Y(t) = 3t^2$

These relationships imply that x and y are functions of t, with linear and quadratic dependencies, respectively. The constraints suggest that x and y are restricted to these forms, perhaps reflecting specific physical or geometric conditions in an application context.

The Function Z and Its Composition

The function Z is given by:

$$
Z = (x + y) e^y
$$

Here, Z depends explicitly on x and y, which are functions of t. To find dz/dt, we must understand how changes in t influence Z via x and y.

Step-by-Step Derivation of dz/dt

Step 1: Express Z in terms of t through $x(t)$ and $y(t)$

Since:

$$
x(t) = 3t
$$
$$
y(t) = 3t^2
$$

we substitute these into Z:

$$
Z(t) = (x(t) + y(t)) e^{y(t)} = (3t + 3t^2) e^{3t^2}
$$

However, directly differentiating this form is possible, but to understand the process thoroughly, it's better to use the chain rule explicitly involving partial derivatives, which is more elegant and generalizable.

Step 2: Compute the partial derivatives $\partial Z / \partial x$ and $\partial Z / \partial y$

Given:

$$
Z = (x + y) e^y
$$

The derivatives are:

- With respect to x:

$$
\frac{\partial Z}{\partial x} = e^y
$$

since y is treated as a constant when differentiating with respect to x.

- With respect to y:

$$
\frac{\partial Z}{\partial y} = (x + y) e^y + e^y
$$

This comes from applying the product rule:

$$
\frac{\partial Z}{\partial y} = \frac{\partial}{\partial y}[(x + y) e^y] = (x + y) \frac{\partial}{\partial y} e^y + e^y \frac{\partial}{\partial y} (x + y)
$$

$$
= (x + y) e^y + e^y (0 + 1) = (x + y) e^y + e^y
$$

which simplifies to:

$$
\frac{\partial Z}{\partial y} = e^y (x + y + 1)
$$

Step 3: Compute dx/dt and dy/dt

From the given functions:

$$
x(t) = 3t \Rightarrow \frac{dx}{dt} = 3
$$

$$
y(t) = 3t^2 \Rightarrow \frac{dy}{dt} = 6t
$$

Step 4: Assemble the total derivative dz/dt using the chain rule

Applying the multivariable chain rule:

$$
\frac{dz}{dt} = \frac{\partial Z}{\partial x} \frac{dx}{dt} + \frac{\partial Z}{\partial y} \frac{dy}{dt}
$$

Substituting the derivatives:

$$
\frac{dz}{dt} = e^y \cdot 3 + e^y (x + y + 1) \cdot 6t
$$

Recall that $x = 3t$ and $y = 3t^2$:

$$
\boxed{
\frac{dz}{dt} = 3 e^{3t^2} + 6t \cdot e^{3t^2} (3t + 3t^2 + 1)
}
$$

which simplifies further to:

$$
\frac{dz}{dt} = e^{3t^2} \left[3 + 6t (3t + 3t^2 + 1) \right]
$$

Analysis of the Derivative Expression

Qualitative Insights

The derivative expression,

$$
\frac{dz}{dt} = e^{3t^2} \left[3 + 6t (3t + 3t^2 + 1) \right],
$$

encapsulates the combined effects of the variables' growth rates on Z. The exponential term e^{3t^2} indicates rapid growth as t increases, reflecting the influence of the quadratic term in the exponent. The polynomial inside the brackets modulates this growth, capturing the linear and quadratic dependencies of x and y.

This structure reveals how the composite function's rate of change is not merely additive but synergistic, with the exponential amplifying the influence of the polynomial terms.

Impact of Restrictions and Assumptions

The problem's restriction that $x = 3t$ and $y = 3t^2$ simplifies the derivative calculation considerably. Without these explicit forms, the derivatives would involve more complex partial derivatives and potentially implicit relationships. The assumptions imply a specific parametric path in the xy-plane, possibly representing a physical trajectory or a specific geometric locus.

Notably, these restrictions allow for direct substitution, converting the multivariable derivative problem into a univariate calculus problem in t, while still respecting the multivariable nature of Z.

Practical Applications and Broader Context

Relevance in Physics and Engineering

Such calculations are fundamental in physics and engineering, where systems evolve over time, and variables are interconnected. For example, in kinematics, position coordinates (x, y) depend on time, and a quantity like Z could represent energy, force, or other physical quantities that depend on position and orientation.

In control systems, understanding how a state variable evolves with respect to time hinges on related derivatives, especially when multiple variables are coupled through nonlinear relationships.

Implications for Mathematical Modeling

Accurate derivative calculations using the chain rule underpin simulations, optimizations, and predictive modeling in various fields. Recognizing the dependencies and correctly applying the chain rule ensures reliable analyses and prevents errors that can cascade into flawed conclusions.

Furthermore, the explicit derivation process enhances understanding of the interplay between variables and parameters, fostering better intuition about system dynamics.

Conclusion: The Power and Precision of the Chain Rule

The detailed derivation of dz/dt for the function $Z = (x + y) e^y$, with $x = 3t$ and $y = 3t^2$, exemplifies the elegance and necessity of the chain rule

If Z X Y E Y X 3t Y 3 T 2 Find Dz Dt Using The Chain Rule Assume The Variables Are Restricted

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