In A Sequence, Each Term After The First Is Four More Than Three Times The Previous Term. The Fifth Term

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Understanding how sequences are constructed and analyzed is fundamental in mathematics. The sequence described here follows a specific recursive rule: each term after the first is obtained by multiplying the previous term by three and then adding four. This type of sequence, known as a linear recurrence relation, is common in various mathematical and real-world applications, such as population modeling, financial calculations, and algorithm analysis. In this article, we will explore this sequence in detail, focusing on how to determine the fifth term, starting from the initial term, and extending the analysis to general terms and properties of the sequence.

Understanding the Sequence: The Recursive Rule

Definition of the Sequence

The sequence $\{a_n\}$ is defined by the recursive relation:

$$
a_{n} = 3a_{n-1} + 4
$$

for $n \geq 2$, with an initial term a_1 given.

This means that to find any term beyond the first, you multiply the previous term by three and then add four.

Initial Term

The sequence's starting point is crucial because it influences all subsequent terms. The initial term a_1 can be any real number, but for the purpose of this exploration, we will consider an initial value $a_1 = a$, where a is a known number.

Calculating the First Few Terms

Step-by-step Computation

Given the recursive formula:

$$
a_{n} = 3a_{n-1} + 4
$$

and initial value $a_1 = a$, we can compute subsequent terms:

1. First term:
$$
a_1 = a
$$

2. Second term:
$$
a_2 = 3a_1 + 4 = 3a + 4
$$

3. Third term:
$$
a_3 = 3a_2 + 4 = 3(3a + 4) + 4 = 9a + 12 + 4 = 9a + 16
$$

4. Fourth term:
$$
a_4 = 3a_3 + 4 = 3(9a + 16) + 4 = 27a + 48 + 4 = 27a + 52
$$

5. Fifth term:
$$
a_5 = 3a_4 + 4 = 3(27a + 52) + 4 = 81a + 156 + 4 = 81a + 160
$$

From these calculations, we observe a pattern emerging in the coefficients and constants.

Deriving a General Formula for the nth Term

Recognizing the Pattern

Looking at the terms:

- $a_1 = a$
- $a_2 = 3a + 4$
- $a_3 = 9a + 16$
- $a_4 = 27a + 52$

- $a_5 = 81a + 160$

We notice that the coefficients of a are powers of 3:

$$
a_1 = 3^0 a
$$
$$
a_2 = 3^1 a
$$
$$
a_3 = 3^2 a
$$
$$
a_4 = 3^3 a
$$
$$
a_5 = 3^4 a
$$

Similarly, the constant terms seem to follow a pattern, which we will analyze next.

Finding a Closed-Form Expression

The recursive relation:

$$
a_{n} = 3a_{n-1} + 4
$$

is a non-homogeneous linear recurrence. To solve it, we use the method of solving homogeneous and particular solutions.

Step 1: Homogeneous Solution

Ignore the constant term:

$$
a_{n}^h = 3a_{n-1}^h
$$

which gives the homogeneous solution:

$$
a_{n}^h = C \cdot 3^{n-1}
$$

where C is a constant determined by initial conditions.

Step 2: Particular Solution

Assuming a constant particular solution $a_{n}^p = A$, substitute into the recurrence:

$$
A = 3A + 4
$$

which simplifies to:

$$
A - 3A = 4 \Rightarrow -2A = 4 \Rightarrow A = -2
$$

Step 3: General Solution

The general solution is:

$$
a_{n} = a_{n}^h + a_{n}^p = C \cdot 3^{n-1} - 2
$$

Use initial condition $a_1 = a$:

$$
a_1 = C \cdot 3^{0} - 2 = C - 2
$$
$$
\Rightarrow C = a + 2
$$

Final formula for the nth term:

$$
\boxed{
a_n = (a + 2) \cdot 3^{n-1} - 2
}
$$

This formula allows us to compute the nth term directly, without recursive calculations.

Applying the Formula to Find the Fifth Term

Given an Initial Term

Suppose the initial term is $a_1 = a$. To find the fifth term a_5:

$$
a_5 = (a + 2) \cdot 3^{4} - 2 = (a + 2) \cdot 81 - 2
$$

Expanding:

$$
a_5 = 81a + 162 - 2 = 81a + 160
$$

which matches our earlier recursive calculation.

Example: Specific Initial Value

If, for instance, $a_1 = 1$:

$$
a_5 = (1 + 2) \times 81 - 2 = 3 \times 81 - 2 = 243 - 2 = 241
$$

Thus, with this initial value, the fifth term is 241.

Understanding the Behavior of the Sequence

Growth Pattern

Since the dominant term involves 3^{n-1}, the sequence grows exponentially if $a \neq -2$. The initial value influences the starting point, but the exponential growth rate is determined by the base 3.

Special Cases

- When $a = -2$, the formula simplifies:

$$
a_n = (-2 + 2) \cdot 3^{n-1} - 2 = 0 - 2 = -2
$$

This implies all terms are equal to -2, meaning the sequence is constant in this special case.

Applications and Importance of Such Sequences

Modeling Real-World Phenomena

Sequences governed by linear recurrence relations like this one can model various situations:

- Population growth with constant additive factors
- Financial calculations involving recursive interest or payments
- Computer algorithms that update values based on previous outputs

Mathematical Significance

Analyzing such sequences introduces students to:

- Recursion
- Closed-form solutions
- Growth behaviors
- Limit analysis

These concepts are foundational in higher mathematics and computer science.

Summary and Key Takeaways

- The sequence is defined recursively by $a_{n} = 3a_{n-1} + 4$, with an initial term $a_1 = a$.
- The first few terms can be calculated step-by-step, but a closed-form formula simplifies this process:

$$
a_n = (a + 2) \cdot 3^{n-1} - 2
$$

- Using this formula, the fifth term is:

  $$
  a_5 = (a + 2) \times 81 - 2
  $$

  which depends on the initial term a.

- Understanding the behavior of the sequence helps in modeling exponential growth and analyzing recursive systems.
- The sequence exhibits exponential growth unless initialized at the special value \(a = -2 \), which results in a constant sequence.

Conclusion

Sequences defined by recursive rules such as "each term after the first is four more than three times the previous term" are critical in understanding recursive relations, growth patterns, and their applications. Deriving a closed-form expression enables quick computation of any term, including the fifth term, without resorting to repetitive calculations. Recognizing these patterns and formulas enhances problem-solving skills and provides insight into the behavior of recursive systems across mathematics and applied sciences.

Frequently Asked Questions

What is the recursive formula for the sequence where each term after the first is four more than three times the previous term?

The recursive formula is $T_n = 3T_{n-1} + 4$, with a given initial term T_1.

If the first term of the sequence is 2, what is the fifth term?

Starting with $T_1=2$, the sequence is $T_2=3(2)+4=10$, $T_3=3(10)+4=34$, $T_4=3(34)+4=106$, $T_5=3(106)+4=322$. So, the fifth term is 322.

Can you find a direct formula to compute the nth term of this sequence?

Yes. Since it's a linear recurrence, the nth term can be expressed as $T_n = A\ 3^{n-1} + B$, where A and B are constants determined by initial conditions.

What is the explicit formula for the sequence if the first term is $T_1=1$?

With $T_1=1$, the explicit formula is $T_n = (1/2)\ 3^n + (1/2)$, derived using solving the recurrence relation.

How do you find the fifth term of the sequence if the first term is unknown?

You need the first term to compute subsequent terms. Once T_1 is known, apply the recursive formula repeatedly to find T_2 through T_5.

Is this sequence an arithmetic or geometric sequence?

No, it's neither. Each term depends on a multiplication and addition, making it a linear recurrence sequence, not arithmetic or geometric.

What real-world situations can be modeled by such a sequence?

This sequence can model scenarios like population growth with specific reproduction rules, or investments with compounded interest plus fixed increments.

How can this sequence be useful in solving related mathematical problems?

Understanding its recursive and explicit forms enables solving for specific terms, sums of terms, or analyzing long-term behavior in various applications.

Additional Resources

Sequence Analysis: Unveiling the Fifth Term in a Recurring Pattern

Understanding sequences and their underlying patterns is a fundamental aspect of mathematical exploration. In this detailed review, we will dissect a specific sequence where each term after the first is defined by the relation: each term is four more than three times the previous term. Our focus will be on the fifth term of this sequence, delving into the mechanics of its calculation, the pattern it follows, and its broader implications in mathematical reasoning.

Introduction to the Sequence Pattern

Sequences are ordered lists of numbers following a particular rule or pattern. The sequence under consideration is characterized by a recursive relation, which means each term depends on its immediate predecessor.

Specifically, the sequence is defined as:

- The first term, a_1, is given or assumed.
- Each subsequent term, a_{n}, satisfies:

$$
a_{n} = 3 \times a_{n-1} + 4
$$

This relation indicates that to find any term, multiply the previous term by 3 and then add 4. This pattern is a linear recurrence relation with constant coefficients, often encountered in algebra and discrete mathematics.

Understanding the Recursive Relation

Definition and Significance

The recursive relation:

$$
a_{n} = 3a_{n-1} + 4
$$

models a process where each step amplifies the previous term by a factor of three and then shifts the value by four. Such sequences are common in various fields, including finance (compound interest models), population dynamics, and algorithmic processes.

The relation’s structure embodies a linear recurrence relation with constant coefficients, which typically produces exponential growth or decay, depending on the initial conditions and the coefficients involved.

Initial Term Specification

To proceed with concrete calculations, an initial term a_1 must be specified. While the problem statement may leave it open, common choices include:

- Default assumption: $a_1 = 1$
- Other possible values: $a_1 = 0$, $a_1 = 2$, or any arbitrary number

For the purpose of this analysis, we will assume $a_1 = 1$, which is a typical starting point and allows us to illustrate the sequence’s behavior clearly.

Calculating the Terms of the Sequence

Starting with $a_1 = 1$, we can calculate subsequent terms step-by-step:

Term Number	Calculation	Result
1	Given $a_1 = 1$	1
2	$a_2 = 3 \times a_1 + 4 = 3 \times 1 + 4 = 7$	7
3	$a_3 = 3 \times a_2 + 4 = 3 \times 7 + 4 = 25$	25
4	$a_4 = 3 \times a_3 + 4 = 3 \times 25 + 4 = 79$	79
5	$a_5 = 3 \times a_4 + 4 = 3 \times 79 + 4 = 241$	241

Thus, the fifth term a_5 in this sequence, assuming an initial term of 1, is 241.

Deriving a Closed-Form Expression

While recursive calculations are straightforward, they can become cumbersome for larger terms. To facilitate direct computation of any term, mathematicians often derive a closed-form formula.

General Solution of the Recurrence Relation

The recurrence:

$$
a_{n} = 3a_{n-1} + 4
$$

is linear and nonhomogeneous. Its general solution involves:

1. Solving the homogeneous part:

$$
a_{n}^{(h)} = r^{n}
$$

with characteristic equation:

$$
r = 3r \Rightarrow r - 3r = 0 \Rightarrow -2r = 0 \Rightarrow r = 0
$$

which indicates the homogeneous solution:

$$
a_{n}^{(h)} = C \times 3^{n-1}
$$

2. Finding a particular solution:

Assuming a constant particular solution $a_{n}^{(p)} = A$, substitute into the recurrence:

$$
A = 3A + 4
$$

$$
A - 3A = 4 \Rightarrow -2A = 4 \Rightarrow A = -2
$$

3. Combine solutions:

$$
a_{n} = a_{n}^{(h)} + a_{n}^{(p)} = C \times 3^{n-1} - 2
$$

4. Determine C using initial conditions:

At $n=1$:

$$
a_1 = C \times 3^{0} - 2 = C - 2
$$

Given $a_1 = 1$:

$$
1 = C - 2 \Rightarrow C = 3
$$

Final closed-form formula:

$$
\boxed{
a_{n} = 3 \times 3^{n-1} - 2
}
$$

which simplifies to:

$$
a_{n} = 3^{n} - 2
$$

Verification:

- For $n=1$:

$$
a_1 = 3^{1} - 2 = 3 - 2 = 1
$$

- For $n=2$:

$$
a_2 = 3^{2} - 2 = 9 - 2 = 7
$$

- For $n=3$:

$$
a_3 = 3^{3} - 2 = 27 - 2 = 25
$$

- For $n=4$:

$$
a_4 = 3^{4} - 2 = 81 - 2 = 79
$$

- For $n=5$:

$$
a_5 = 3^{5} - 2 = 243 - 2 = 241
$$

which confirms our previous recursive calculation.

Implications and Broader Insights

Growth Rate and Pattern

From the closed-form expression:

$$
a_{n} = 3^{n} - 2
$$

we observe exponential growth governed by the base 3. Each subsequent term is approximately three times larger than the previous one, minus a small adjustment (subtracting 2).

This rapid increase illustrates how recursive sequences with multiplicative factors can model exponential phenomena, making them relevant in contexts like population modeling, investment growth, or the spread of information.

Applications in Real-World Scenarios

Sequences with similar recursive definitions appear in various fields:

- Computer Science: Algorithm complexity analysis, especially recursive algorithms.
- Biology: Population growth models where reproduction multiplies the population.
- Finance: Compound interest calculations with fixed contributions.
- Physics: Radioactive decay or chain reactions modeled via recursive relations.

Understanding the fifth term in such sequences provides critical insight into the long-term behavior of the modeled system.

Summary and Key Takeaways

- The sequence is defined by the recursive relation:

$$
a_{n} = 3a_{n-1} + 4
$$

- Assuming $a_1 = 1$, the sequence progresses as: 1, 7, 25, 79, 241, ...
- The fifth term, a_5, is 241.
- The closed-form formula simplifies calculations for any term:

$$
a_{n} = 3^{n} - 2
$$

- The sequence exhibits exponential growth, with the terms increasing approximately threefold each step.

- Recognizing such patterns enhances problem-solving skills in mathematics and related disciplines.

Conclusion: The Significance of Pattern Recognition

Analyzing the sequence and determining its fifth term exemplifies the power of recursive relations and their closed-form solutions. Such analytical tools allow mathematicians and scientists to predict future terms, understand growth behaviors, and apply these insights across various fields. Whether modeling biological populations, financial investments, or computational processes, mastering the mechanics behind these sequences is essential for advancing both theoretical understanding and practical applications.

By dissecting this particular sequence, we've illuminated the elegance of recursive definitions and their exponential nature, highlighting the importance of foundational concepts like recurrence relations, characteristic equations, and closed-form formulas in mathematical analysis.

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