Let A Be An N X N Matrix With An Eigenvalue Of Multiplicity N. Show That A Is Diagonalizable If And Only

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Introduction

Understanding the conditions under which a matrix is diagonalizable is a fundamental aspect of linear algebra. Diagonalization simplifies many matrix computations, including powers, functions of matrices, and solving differential equations. When analyzing an N x N matrix A, the nature of its eigenvalues and their algebraic and geometric multiplicities plays a crucial role in determining whether A can be represented as a diagonal matrix in some basis.

In this article, we focus on the special case where A has a single eigenvalue λ with algebraic multiplicity N. We aim to explore and establish the necessary and sufficient conditions for A to be diagonalizable in this scenario. The core question addressed is: Given that the eigenvalue λ has multiplicity N, under what conditions is A diagonalizable?

Preliminaries and Key Concepts

Before delving into the main discussion, it is essential to revisit some fundamental definitions and concepts related to eigenvalues, eigenvectors, and diagonalization.

Eigenvalues and Eigenvectors

- Eigenvalue (λ): A scalar λ is an eigenvalue of matrix A if there exists a non-zero vector v such that:

$$
A v = \lambda v
$$

- Eigenvector: The non-zero vector v satisfying the above equation for a given eigenvalue λ.

Algebraic and Geometric Multiplicity

- Algebraic Multiplicity: The multiplicity of λ as a root of the characteristic polynomial $\det(A - \lambda I) = 0$.

- Geometric Multiplicity: The dimension of the eigenspace associated with λ, i.e., $\dim(\ker(A - \lambda I))$.

Diagonalizability

A matrix A is diagonalizable over a field (e.g., real or complex numbers) if there exists an invertible matrix P such that:

$$
A = P D P^{-1}
$$

where D is a diagonal matrix containing the eigenvalues of A.

Key fact: A matrix is diagonalizable if and only if the sum of the dimensions of its eigenspaces equals N, the size of the matrix. Equivalently, the minimal polynomial splits into distinct linear factors, and the geometric multiplicity of each eigenvalue matches its algebraic multiplicity.

Scenario: A Has a Single Eigenvalue λ with Multiplicity N

Given that A has an eigenvalue λ with algebraic multiplicity N, the characteristic polynomial takes the form:

$$
\chi_A(\lambda) = (\lambda - \lambda_0)^N
$$

for some $\lambda_0 \in \mathbb{C}$ (or $\mathbb{R}$ if over the reals). Our goal is to analyze the conditions under which A is diagonalizable in this context.

Diagonalizability in the Case of a Single Eigenvalue

Since A has only one eigenvalue λ, its diagonalizability hinges entirely on the geometric multiplicity of λ.

Key Criterion

- A is diagonalizable if and only if the geometric multiplicity of λ is N.

This is because:

- The algebraic multiplicity is N, and
- The sum of the dimensions of all eigenspaces (here just one eigenspace) must be N for diagonalizability.

The main challenge is to determine when the geometric multiplicity equals N.

Necessary and Sufficient Conditions for Diagonalizability

Let's formalize the conditions:

Necessity

Suppose A is diagonalizable. Then:

$$
A = P D P^{-1}
$$

with D diagonal. Since A has only one eigenvalue λ, D must be a scalar multiple of the identity matrix:

$$
D = \lambda I
$$

Therefore, the eigenvalues are all λ, and the geometric multiplicity of λ (the dimension of the eigenspace) must be N. This implies that:

$$
\dim(\ker(A - \lambda I)) = N
$$

which means:

$$
A - \lambda I = 0
$$

or, more generally, that the eigenspace associated with λ spans the entire space $\mathbb{C}^N$. So, the necessary condition is:

$$
A = \lambda I
$$

or, at a minimum, that the eigenspace has dimension N.

Sufficiency

Conversely, if the geometric multiplicity of λ is N, then:

$$
\dim(\ker(A - \lambda I)) = N
$$

which implies that:

$$
A - \lambda I = 0
$$

or, in general, that all vectors are eigenvectors corresponding to λ. In this case, A is similar to λI, which is clearly diagonal (a scalar multiple of the identity).

Thus, the sufficiency condition becomes:

$$
\text{Geometric multiplicity of } \lambda = N
$$

which guarantees diagonalizability.

Summary of the Main Result

Theorem:

Let A be an N x N matrix with a single eigenvalue λ of algebraic multiplicity N. Then, A is diagonalizable if and only if the geometric multiplicity of λ is N.

Implication:

- If the geometric multiplicity of λ is N, then A is similar to λI, i.e., A is a scalar matrix or similar to a scalar matrix.
- If the geometric multiplicity is less than N, then A is not diagonalizable; it has Jordan blocks of size greater than one.

Jordan Normal Form Perspective

To deepen our understanding, consider the Jordan normal form. Every matrix A over $\mathbb{C}$ (or algebraically closed fields) can be expressed as:

$$
A = P J P^{-1}
$$

where J is a block-diagonal matrix with Jordan blocks.

In our case, since the only eigenvalue is λ, the Jordan form J looks like:

$$
J = \bigoplus_{i=1}^{k} J_{m_i}(\lambda)
$$

where each $J_{m_i}(\lambda)$ is a Jordan block of size m_i.

For diagonalizability:

- All Jordan blocks are of size 1, i.e.,

$$
J = \lambda I
$$

- The sum of sizes of the Jordan blocks equals N.

Hence, diagonalizability is equivalent to all Jordan blocks being size 1, which occurs precisely when the geometric multiplicity equals the algebraic multiplicity.

Examples Illustrating the Conditions

Example 1: Diagonalizable Case

Suppose:

$$
A = \lambda I
$$

- The eigenvalue λ has algebraic multiplicity N.
- The eigenspace is $\mathbb{C}^N$, so geometric multiplicity is N.
- A is diagonal and clearly diagonalizable.

Example 2: Non-Diagonalizable Case

Let:

$$
A = \begin{bmatrix}
\lambda & 1 \\
0 & \lambda
\end{bmatrix}
$$

- The characteristic polynomial is $(\lambda - \lambda)^2 = 0$, so algebraic multiplicity is 2.
- The eigenspace is spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so geometric multiplicity is 1.
- Since geometric multiplicity < algebraic multiplicity, A is not diagonalizable.

Practical Implications and Applications

Understanding the conditions under which a matrix with a single eigenvalue is diagonalizable has practical consequences in various fields:

- Simplifying matrix functions: When A is diagonalizable, functions of A (like exponentials) are easier to compute.
- Solving differential equations: Diagonalization aids in solving systems of linear

differential equations with constant coefficients.
- Quantum mechanics: Diagonal operators correspond to observable quantities, and understanding when operators are diagonalizable informs measurement theory.
- Numerical linear algebra: Recognizing when matrices are diagonalizable helps in designing stable algorithms.

Summary and Final Remarks

In summary, for an N x N matrix A with a single eigenvalue λ of algebraic multiplicity N:

- A is diagonalizable if and only if the geometric multiplicity of λ is N.
- Equivalently, A must be similar to λI, i.e., a scalar multiple of the identity matrix, or at least have a complete basis of eigenvectors spanning $\mathbb{C}^N$.

This criterion underscores the importance of the geometric multiplicity in the context of eigenvalues and diagonalization. When the eigenvalue's geometric multiplicity matches its algebraic multiplicity, the matrix admits a basis of eigenvectors, enabling a diagonal representation.

References and Further Reading

- Lay, David C., "Linear Algebra and Its Applications," 4th Edition.
- Hoffman, Kenneth and Kunze, Ray, "Linear Algebra," 2nd Edition.
- Axler

Frequently Asked Questions

What does it mean for an n x n matrix A to have an eigenvalue of multiplicity n?

It means that there is a single eigenvalue λ such that its algebraic multiplicity, the multiplicity as a root of the characteristic polynomial, is n. In this case, λ is the only eigenvalue of A, with multiplicity n.

What is the significance of an eigenvalue having algebraic multiplicity n for the matrix A?

It indicates that the characteristic polynomial is $(\lambda - \lambda_0)^n$, and that λ_0 is the only eigenvalue. The matrix's eigenstructure is heavily centered around this single eigenvalue, affecting its diagonalizability.

Under what condition is a matrix with a single eigenvalue diagonalizable?

A matrix with a single eigenvalue is diagonalizable if and only if its geometric multiplicity equals its algebraic multiplicity, meaning the eigenspace has dimension n.

How does the geometric multiplicity relate to the diagonalizability of matrix A in this context?

Since A has only one eigenvalue with algebraic multiplicity n, A is diagonalizable if and only if the dimension of the eigenspace corresponding to that eigenvalue is n, i.e., the eigenspace is n-dimensional.

What is the 'if and only if' condition for A to be diagonalizable when it has an eigenvalue of multiplicity n?

A is diagonalizable if and only if the geometric multiplicity of the eigenvalue equals n, meaning the eigenspace spans the entire space R^n.

Can a matrix with a single eigenvalue and algebraic multiplicity n fail to be diagonalizable?

Yes, if the geometric multiplicity of the eigenvalue is less than n, then the matrix is not diagonalizable because it lacks a complete basis of eigenvectors.

What is the connection between Jordan canonical form and the diagonalizability of matrix A in this case?

If the matrix's Jordan form consists solely of n 1x1 blocks (i.e., it is diagonal), then A is diagonalizable. Otherwise, if there are Jordan blocks larger than 1x1, A is not diagonalizable.

How can one verify whether A is diagonalizable given it has a single eigenvalue with algebraic multiplicity n?

Compute the dimension of the eigenspace (geometric multiplicity). If it equals n, A is diagonalizable; if less than n, it is not.

Summarize the necessary and sufficient condition for A to be diagonalizable in this scenario.

A is diagonalizable if and only if the geometric multiplicity of the eigenvalue equals its algebraic multiplicity n, i.e., the eigenspace is n-dimensional, which ensures the existence of a basis of eigenvectors for R^n.

Additional Resources

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Introduction

Eigenvalues and diagonalizability are fundamental concepts in linear algebra, underpinning much of modern mathematics, physics, and engineering. When analyzing an $N \times N$ matrix A, understanding its eigenstructure provides insights into its behavior, especially regarding spectral decomposition and matrix functions. The scenario where A has an eigenvalue λ with algebraic multiplicity N — the size of the matrix — presents a particularly interesting case. Such matrices are closely related to scalar matrices but may exhibit nuanced properties depending on their eigenvectors and eigenprojections.

This article delves into the conditions under which such a matrix A is diagonalizable, emphasizing the necessary and sufficient conditions. The core question addressed is: "Given an $N \times N$ matrix A with an eigenvalue λ of multiplicity N, under what circumstances is A diagonalizable?" The exploration offers a detailed analysis, blending algebraic and geometric perspectives, with rigorous explanations suitable for both students and researchers.

Background and Fundamental Concepts

Eigenvalues and Eigenvectors

An eigenvalue λ of an $N \times N$ matrix A satisfies the characteristic equation:

$$
\det(A - \lambda I) = 0
$$

The eigenvalues are roots of the characteristic polynomial, which is of degree N. For each eigenvalue λ, the geometric multiplicity is defined as the dimension of the eigenspace:

$$
\operatorname{dim}(\ker(A - \lambda I))
$$

The algebraic multiplicity of λ is its multiplicity as a root of the characteristic polynomial.

Diagonalizability and Its Criteria

A matrix A is diagonalizable over a field (typically $\mathbb{R}$ or $\mathbb{C}$) if it is similar to a diagonal matrix, i.e., there exists an invertible matrix P such that:

$$
A = P D P^{-1}
$$

where D is a diagonal matrix containing the eigenvalues of A.

Key criterion for diagonalizability:

- A is diagonalizable iff the sum of the geometric multiplicities of its eigenvalues equals N, i.e., the sum over all eigenvalues of their geometric multiplicities is N.
- Equivalently, A is diagonalizable iff the sum of the algebraic multiplicities equals N, and for each eigenvalue, the geometric multiplicity equals its algebraic multiplicity.

The Special Case: Single Eigenvalue With Full Multiplicity

The Setup

Suppose A is an $N \times N$ matrix with a single eigenvalue:

$$
\lambda \quad \text{with algebraic multiplicity} \quad N
$$

This implies:

$$
\det(A - \lambda I) = 0
$$

has degree N, and λ is the only root. The characteristic polynomial simplifies to:

$$
p_A(\mu) = (\mu - \lambda)^N
$$

The matrix A is therefore cyclically similar to a Jordan block (or a direct sum of Jordan blocks) associated with λ.

Eigenvalue Multiplicity and Diagonalizability

The question focuses on whether such a matrix A is necessarily diagonalizable. The answer hinges on the geometric multiplicity:

- If the geometric multiplicity equals N, then A has N linearly independent eigenvectors and is diagonalizable.
- If the geometric multiplicity is less than N, then A cannot be diagonalized, as it lacks sufficient eigenvectors to form a basis.

In essence:

$$
\text{Diagonalizability} \iff \text{geometric multiplicity of } \lambda = N
$$

Analytical Framework and Theorem Statement

Formal Theorem

> Theorem:
> Let $A \in \mathbb{C}^{N \times N}$ (or over $\mathbb{R}$, provided the eigenvalues are complex). Suppose A has an eigenvalue λ with algebraic multiplicity N. Then, A is diagonalizable if and only if the geometric multiplicity of λ is N.

In words:

> The matrix A is diagonalizable if and only if it has N linearly independent eigenvectors corresponding to λ.

Detailed Explanation and Proof

Necessity: If A Is Diagonalizable, Then Geometric Multiplicity Is N

Assuming A is diagonalizable, then there exists an invertible matrix P such that:

$$
A = P D P^{-1}
$$

where D is a diagonal matrix with λ on the diagonal, repeated N times:

$$
D = \lambda I
$$

Therefore, the eigenvectors are columns of P, and the eigenspace for λ is spanned by these vectors. Because P is invertible, these eigenvectors are linearly independent, and there are N of them:

$$
\operatorname{dim}(\ker(A - \lambda I)) = N
$$

Thus, the geometric multiplicity of λ must be N.

Sufficiency: If the Geometric Multiplicity Is N, Then A Is Diagonalizable

Suppose that the geometric multiplicity of λ is N. This implies:

$$
\ker(A - \lambda I) = \mathbb{C}^N
$$

or, in real matrices, the entire space is spanned by eigenvectors associated with \(\lambda

\). Since all eigenvectors correspond to the same eigenvalue, and their number equals N, they form a basis for the entire space:

$$
\text{Eigenvector basis:} \quad \{ v_1, v_2, \ldots, v_N \} \quad \text{with} \quad (A - \lambda I) v_i = 0
$$

Constructing the matrix P with these eigenvectors as columns yields:

$$
A P = P \lambda I
$$

or

$$
A = P \lambda I P^{-1} = \lambda I
$$

which is already diagonal. Therefore, A is similar to a diagonal matrix with λ on the diagonal—the scalar matrix λI—which is trivially diagonal.

Special Cases and Additional Considerations

Multiple Eigenvalues and Diagonalizability

While the core focus is on a single eigenvalue with multiplicity N, it’s instructive to consider more nuanced cases:

- Diagonalizable matrices with a single eigenvalue: These are scalar matrices $A = \lambda I$.
- Non-diagonalizable matrices with a single eigenvalue: These are matrices with Jordan blocks of size greater than 1, e.g., nilpotent matrices like nilpotent Jordan blocks.

Implications of the Jordan Normal Form

The Jordan normal form provides a canonical form for matrices over algebraically closed fields:

$$
A \sim J = P^{-1} A P
$$

where J is block diagonal with Jordan blocks. For the case where all eigenvalues are λ, the Jordan form looks like:

$$
J = \lambda I + N
$$

where N is a nilpotent matrix with blocks corresponding to Jordan blocks. The size of these blocks is directly related to the geometric multiplicity:

- All blocks of size 1: $J = \lambda I$, the matrix is diagonal and diagonalizable.
- At least one block larger than size 1: J is not diagonal, and A is not diagonalizable.

Hence, the geometric multiplicity corresponds to the number of Jordan blocks, and the algebraic multiplicity corresponds to the sum of their sizes.

Concluding Remarks

Summary of Key Results:

1. If an $N \times N$ matrix A has an eigenvalue λ with algebraic multiplicity N, then A is a single Jordan block or a sum of blocks with eigenvalue λ.

2. The matrix A is diagonalizable if and only if the geometric multiplicity of \(\lambda \

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