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Introduction to Vector Spaces, Bases, and Linear Transformations

Understanding the fundamental concepts of vector spaces, bases, and linear transformations is crucial in the study of linear algebra. These concepts form the backbone of many areas in mathematics, physics, engineering, and computer science, providing tools to analyze multidimensional data, solve systems of equations, and model complex systems.

A vector space is a collection of objects called vectors, which can be added together and multiplied by scalars, satisfying specific axioms such as associativity, commutativity of addition, distributivity, and the existence of additive identities and inverses. Examples include Euclidean space $\mathbb{R}^n$, function spaces, and polynomial spaces.

A basis of a vector space is a set of vectors that are linearly independent and span the entire space. The basis provides a way to uniquely represent any vector in the space as a linear combination of the basis vectors.

A linear transformation $T: V \to W$ is a function between vector spaces that preserves vector addition and scalar multiplication:
- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
- $T(c\mathbf{v}) = cT(\mathbf{v})$

These properties make linear transformations analogous to matrices, allowing us to analyze and compute transformations efficiently.

Understanding the Given Problem

The problem states:

"Let B = {b1, b2, b3} be a basis for vector space V. Find T(3b1 - 4b2) when T is a linear transformation from V to another vector space."

This problem involves several key components:

- The basis $B = \{b_1, b_2, b_3\}$, which spans the vector space V.
- A specific vector $3b_1 - 4b_2$ in V expressed as a linear combination of basis vectors.
- A linear transformation $T : V \to W$, where W is another vector space.
- The task is to find the image $T(3b_1 - 4b_2)$.

To solve this problem, we need to understand how linear transformations act on vectors, especially those expressed in a basis, and how they can be computed once their effect on basis vectors is known.

Key Concepts for Solving the Problem

1. Representation of Vectors in a Basis

Any vector $v \in V$ can be expressed as a linear combination of the basis vectors:

$$
v = a_1b_1 + a_2b_2 + a_3b_3
$$

where a_1, a_2, a_3 are scalars.

For the specific vector $3b_1 - 4b_2$, the coefficients are:

$$
a_1 = 3,\quad a_2 = -4,\quad a_3 = 0
$$

2. Linear Transformation and Its Action on Basis Vectors

A linear transformation $T: V \to W$ is fully determined by its action on a basis of V. That is, if we know:

$$
T(b_1),\quad T(b_2),\quad T(b_3)
$$

then we can find $T(v)$ for any $v \in V$ by linearity:

$$
T(v) = a_1 T(b_1) + a_2 T(b_2) + a_3 T(b_3)
$$

3. Computing $T(3b_1 - 4b_2)$

Using linearity:

$$
T(3b_1 - 4b_2) = 3 T(b_1) - 4 T(b_2)
$$

since T is linear, the image of a linear combination is the linear combination of the images.

Step-by-Step Solution Approach

To find $T(3b_1 - 4b_2)$, follow these steps:

1. Determine or Obtain $T(b_1)$ and $T(b_2)$

In most problems like this, the images of basis vectors under T are given or can be deduced. If the problem provides:
- $T(b_1) = w_1$,
- $T(b_2) = w_2$,
- $T(b_3) = w_3$,

then the calculation proceeds directly.

If these are not given, the problem might specify the transformation explicitly or give a matrix representation of T.

2. Use Linearity to Compute $T(3b_1 - 4b_2)$

Apply the linearity property:

$$
T(3b_1 - 4b_2) = 3 T(b_1) - 4 T(b_2)
$$

3. Simplify the Expression

Perform scalar multiplication and vector subtraction in the codomain space W, resulting in the image vector.

Additional Considerations and Special Cases

Case 1: Explicit \(T\) Defined by a Matrix

Suppose \(T\) is represented by a matrix \(A\) with respect to bases \(B\) and some basis \(C\) in \(W\). If:
\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
\vdots & \vdots & \vdots \\
a_{m1} & a_{m2} & a_{m3}
\end{bmatrix}
\]
then:
\[
T(b_j) = \text{the } j^\text{th} \ column \text{ of } A
\]

and:
\[
T(3b_1 - 4b_2) = 3 T(b_1) - 4 T(b_2)
\]
which can be computed directly.

Case 2: \(T\) is Not Fully Specified

If the problem doesn’t specify \(T(b_1)\) and \(T(b_2)\), then the best you can do is express the answer in terms of these images:
\[
T(3b_1 - 4b_2) = 3 T(b_1) - 4 T(b_2)
\]

Example: Practical Calculation

Suppose the problem gives:
\[
T(b_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}
,\quad T(b_2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
\]
and \(T(b_3)\) is irrelevant for this calculation.

Then:
\[

T(3b_1 - 4b_2) = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 12 \\ 16 \end{bmatrix} = \begin{bmatrix} -9 \\ -10 \end{bmatrix}
\]

This example illustrates how to compute the image once the transformations of the basis vectors are known.

Applications and Importance of the Concept

Understanding how to compute \(T(v)\) for vectors expressed in a basis is fundamental in many applications:

- Coordinate transformations: Changing between different coordinate systems in physics and engineering.
- Diagonalization: Finding eigenvalues and eigenvectors involves linear transformations and their action on basis vectors.
- Computer graphics: Transforming geometric objects via matrices representing linear transformations.
- Data science: Dimensionality reduction techniques like Principal Component Analysis (PCA) use linear algebra concepts extensively.
- Quantum mechanics: State transformations are represented by linear operators acting on basis vectors.

Summary and Key Takeaways

- Any vector in a finite-dimensional vector space can be expressed as a linear combination of a basis.
- A linear transformation is fully characterized by its action on basis vectors.
- To find the image of a vector under a linear transformation, express it in terms of basis vectors and then use linearity.
- The process involves simple scalar-vector multiplications and vector additions/subtractions.
- When the transformation's effect on basis vectors is known, computations become straightforward.

Conclusion

The problem of finding $T(3b_1 - 4b_2)$ hinges on understanding the linearity of the transformation and knowing how T acts on the basis vectors. Once these actions are known, the problem reduces to basic algebraic operations. This

Frequently Asked Questions

Given B = {b1, b2, b3} as a basis for V and T is a linear transformation, how do you find T(3b1 - 4b2)?

To find T(3b1 - 4b2), apply linearity: T(3b1 - 4b2) = 3T(b1) - 4T(b2). You need the images of b1 and b2 under T to compute this.

What is the importance of knowing T's action on basis vectors in determining T(3b1 - 4b2)?

Since T is linear, knowing T(b1) and T(b2) allows you to compute T(3b1 - 4b2) directly using linear combinations, making the problem straightforward.

If T(b1) = v1 and T(b2) = v2, how do you express T(3b1 - 4b2)?

T(3b1 - 4b2) = 3v1 - 4v2, based on the linearity property of the transformation.

Can T(3b1 - 4b2) be determined without knowing T(b1) and T(b2)?

No, without knowing T(b1) and T(b2), you cannot explicitly compute T(3b1 - 4b2) because the transformation's effect on basis vectors is necessary.

How does the choice of basis B affect the computation of T(3b1 - 4b2)?

The basis B defines the vectors' representations and the images T(b1) and T(b2). Different bases may change the coordinate expressions but not the linearity property, so knowing T on basis vectors remains key.

Is T(3b1 - 4b2) dependent on the basis B beyond

T(b1) and T(b2)?

No, once T(b1) and T(b2) are known, the specific basis doesn't affect the calculation of T(3b1 - 4b2) due to linearity.

What would happen if T is not linear? Could you find T(3b1 - 4b2)?

If T is not linear, you cannot simply apply linearity to find T(3b1 - 4b2); you'd need the explicit form of T to compute the image.

How can this problem help in understanding the properties of linear transformations?

It demonstrates that linear transformations are fully determined by their action on basis vectors, making the computation of images of other vectors systematic and straightforward.

What additional information is needed to compute T(3b1 - 4b2) explicitly?

You need to know the images T(b1) and T(b2) under the transformation T to explicitly compute T(3b1 - 4b2).

Additional Resources

Let B = {b1, b2, b3} Be A Basis For Vector Space V. Find T(3b1 - 4b2) When T Is A Linear Transformation From

Understanding the behavior of linear transformations with respect to a basis is fundamental in linear algebra. This article explores a specific problem: given a basis $B = \{b_1, b_2, b_3\}$ for a vector space V, and a linear transformation $T: V \to W$, how can we determine the image of a particular vector, namely $3b_1 - 4b_2$? The problem is a classic example illustrating the power of linearity and the importance of knowing how T acts on basis vectors. We will systematically analyze this problem, delve into the concepts involved, and demonstrate methods to find $T(3b_1 - 4b_2)$.

The Foundation: Basis and Linear Transformations

What Is a Basis?

A basis $B = \{b_1, b_2, b_3\}$ of a vector space V is a set of vectors that are:

- Linearly Independent: No vector in the set can be expressed as a linear combination of the others.
- Spanning V: Every vector in V can be expressed as a linear combination of the basis vectors.

Because of these properties, any vector $v \in V$ can be uniquely written as:

$$
v = \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3,
$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{F}$ (the underlying field, such as $\mathbb{R}$ or $\mathbb{C}$).

What Is a Linear Transformation?

A linear transformation $T: V \to W$ is a function satisfying two properties for all vectors $u, v \in V$ and scalar c:

1. Additivity: $T(u + v) = T(u) + T(v)$,
2. Homogeneity: $T(c v) = c T(v)$.

Linear transformations are the backbone of linear algebra, enabling the mapping of vectors from one space to another while preserving vector addition and scalar multiplication.

The Core Problem: Finding $T(3b_1 - 4b_2)$

Given the basis $B = \{b_1, b_2, b_3\}$, the problem asks: if T is a linear transformation from V to some vector space W, what is $T(3b_1 - 4b_2)$?

To solve this, we need to leverage the linearity of T and the known images of basis vectors. The key steps involve:

- Expressing the target vector in terms of basis vectors.
- Applying the linearity property of T.
- Using known T-images of basis vectors, if provided.

Step-by-Step Approach to the Solution

1. Understand the Given Data

The problem typically provides or implies:

- The basis $B = \{b_1, b_2, b_3\}$ of V.

- The images of the basis vectors under T, i.e., $T(b_1)$, $T(b_2)$, and $T(b_3)$.

If these images are not explicitly given, the problem cannot be fully resolved unless further information is provided. For illustration, suppose:

$$
T(b_1) = w_1, \quad T(b_2) = w_2, \quad T(b_3) = w_3,
$$
where $w_1, w_2, w_3 \in W$.

2. Express the Target Vector as a Linear Combination

The vector $3b_1 - 4b_2$ is already expressed as a linear combination of the basis vectors:

$$
v = 3b_1 - 4b_2.
$$

Note that the coefficients are 3 for b_1, -4 for b_2, and implicitly 0 for b_3.

3. Use Linearity to Compute the Image

Since T is linear:

$$
T(v) = T(3b_1 - 4b_2) = 3 T(b_1) - 4 T(b_2).
$$

This step is crucial because linear transformations respect scalar multiplication and addition.

4. Substitute Known Images

If we know $T(b_1) = w_1$ and $T(b_2) = w_2$, then:

$$
T(3b_1 - 4b_2) = 3w_1 - 4w_2.
$$

This result is in the space W. The exact form depends on the images of the basis vectors.

Examples and Illustrations

Let's explore a concrete example to clarify this process.

Suppose:

$$
T(b_1) = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad T(b_2) = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad T(b_3) = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}.
$$

What is $T(3b_1 - 4b_2)$?

Step-by-step:

$$
T(3b_1 - 4b_2) = 3 T(b_1) - 4 T(b_2) = 3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.
$$

Calculate each component:

$$
3 T(b_1) = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix},
$$
$$
4 T(b_2) = \begin{bmatrix} 0 \\ 4 \\ -4 \end{bmatrix}.
$$

Subtract:

$$
T(3b_1 - 4b_2) = \begin{bmatrix} 3 - 0 \\ 0 - 4 \\ 6 - (-4) \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 10 \end{bmatrix}.
$$

Thus, the image of $3b_1 - 4b_2$ under T is $\boxed{\begin{bmatrix} 3 \\ -4 \\ 10 \end{bmatrix}}$.

Generalization and Implications

When the Images of Basis Vectors Are Unknown

If the images $T(b_i)$ are not given explicitly, the problem becomes more abstract. In such cases, the key is to understand the structure of T:

- For example, if T is represented by a matrix A relative to basis B, then:

$$
T(v) = A \cdot \alpha,
$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)^T$ are the coordinates of v in basis B.

- To find $T(3b_1 - 4b_2)$, we need the coordinate vector:

$$
\alpha = \begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}.
$$

- The result is:

$$
T(3b_1 - 4b_2) = A \times \alpha,
$$

which requires the matrix A.

The Importance of Basis Choice

The choice of basis influences the ease of computation. When working with standard bases, the matrix representation of T is straightforward. When working with arbitrary bases, coordinate transformations are necessary.

Extending the Discussion: Applications in Science and Engineering

Understanding how to compute $T(3b_1 - 4b_2)$ is not just an academic exercise; it has real-world applications:

- Computer Graphics: Transformations of vectors (scaling, rotation, translation) often involve basis changes.
- Data Science: Dimensionality reduction involves linear transformations, where basis changes simplify data structure.
- Quantum Mechanics: State transformations rely on understanding linear operators relative to basis vectors.
- Engineering: Signal processing techniques depend on linear transformations and basis representations.

Summary and Takeaways

- The problem of finding $T(3b_1 - 4b_2)$ hinges on the linearity of T.
- Expressing the target vector as a linear combination of basis vectors simplifies the computation.
- Knowing the images of basis vectors under T allows straightforward calculation.
- The problem demonstrates the fundamental principle that linear transformations are determined entirely by their action on a basis.

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In essence, mastering these concepts equips students and professionals with
the tools to analyze complex linear systems efficiently.
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