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Introduction to Polynomial Spaces and Linear Transformations

In the study of linear algebra, understanding the behavior of linear transformations between different vector spaces is fundamental. When these spaces are spaces of polynomials, such as P_2, the set of all polynomials of degree at most 2, the concepts become particularly interesting and applicable across various mathematical and engineering disciplines.

This article explores a specific linear transformation T from the polynomial space P_2 to the real numbers $\mathbb{R}$, defined by the rule:

$$T(p) = [p(0), p(1)]$$

where $p(t) \in P_2$. We will analyze this transformation, find its matrix representation, determine its kernel and image, and understand its properties thoroughly.

Understanding the Polynomial Space P_2

Definition and Basis of P_2

The space P_2 consists of all polynomials with real coefficients of degree less than or equal to 2:

$$P_2 = \{ p(t) = a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R} \}$$

A common basis for P_2 is:

$$\mathcal{B} = \{ 1, t, t^2 \}$$

Any polynomial $p(t) \in P_2$ can be uniquely expressed as a linear combination:

$$ p(t) = a_0 \cdot 1 + a_1 \cdot t + a_2 \cdot t^2 $$

where $a_0, a_1, a_2 \in \mathbb{R}$.

Definition of the Transformation T

The linear transformation $T : P_2 \to \mathbb{R}^2$ is given by:

$$ T(p) = [p(0), p(1)] $$

This transformation takes a polynomial and returns a 2-dimensional vector consisting of the polynomial's value at $t=0$ and $t=1$.

Note: Although the initial statement might suggest the codomain is $\mathbb{R}$, the notation $[p(0), p(1)]$ indicates the output is in $\mathbb{R}^2$. For clarity, we proceed assuming the transformation maps into $\mathbb{R}^2$.

Analyzing T: Action on Basis Polynomials

To understand T, we analyze its effect on the basis elements of P_2:

- For $p(t) = 1$:

$$ T(1) = [1(0), 1(1)] = [1, 1] $$

- For $p(t) = t$:

$$ T(t) = [t(0), t(1)] = [0, 1] $$

- For $p(t) = t^2$:

$$ T(t^2) = [t^2(0), t^2(1)] = [0, 1] $$

These images help us construct the matrix representation of T.

Matrix Representation of T

Given the basis $\mathcal{B} = \{ 1, t, t^2 \}$, the matrix $[T]_\mathcal{B}$ with respect to this basis and the standard basis in $\mathbb{R}^2$ is formed by applying T to each basis vector and expressing the results as columns.

Columns:

- $T(1) = [1, 1]$
- $T(t) = [0, 1]$
- $T(t^2) = [0, 1]$

Matrix:

$$
[T]_\mathcal{B} = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
$$

This matrix acts on the coordinate vector $[a_0, a_1, a_2]^T$ of a polynomial $p(t)$ in the basis $\{1, t, t^2\}$, producing a vector in $\mathbb{R}^2$:

$$
T(p) = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2
\end{bmatrix}
$$

Explicit Formula for $T(p)$

Expressing $p(t)$ explicitly:

$$ p(t) = a_0 + a_1 t + a_2 t^2 $$

then:

$$
T(p) = \left[p(0), p(1) \right] = \left[a_0, a_0 + a_1 + a_2 \right]
$$

which can be directly obtained from the coefficients:

$$
T(p) = \left(a_0, a_0 + a_1 + a_2 \right)
$$

Alternatively, in terms of the polynomial's coefficients:

$$
T(p) = \left(a_0, a_0 + a_1 + a_2 \right)
$$

This explicit form aids in understanding the linearity and other properties of T.

Properties of the Transformation T

Linearity

Since T is defined via evaluation at points and addition/scaling of polynomials, it is linear:

$$
T(p + q) = T(p) + T(q)
\quad \text{and} \quad
T(\alpha p) = \alpha T(p)
$$

for all $p, q \in P_2$ and $\alpha \in \mathbb{R}$.

Kernel of T

The kernel (null space) of T:

$$
\ker(T) = \{ p \in P_2 \mid T(p) = [0, 0] \}
$$

which implies:

$$
p(0) = 0, \quad p(1) = 0
$$

Since $p(t) = a_0 + a_1 t + a_2 t^2$, these conditions give:

$$

$$
a_0 = 0
$$
$$
a_0 + a_1 + a_2 = 0 \implies a_1 + a_2 = 0
$$

Thus, the kernel consists of all polynomials:

$$
p(t) = 0 + a_1 t + a_2 t^2
$$

with:

$$
a_1 + a_2 = 0 \implies a_2 = -a_1
$$

Expressed as:

$$
p(t) = a_1 t - a_1 t^2 = a_1 (t - t^2)
$$

where $a_1 \in \mathbb{R}$. Therefore, the kernel is:

$$
\ker(T) = \text{span}\{ t - t^2 \}
$$

which is a one-dimensional subspace of P_2.

Image of T

The image (range) is:

$$
\operatorname{Im}(T) = \{ (a_0, a_0 + a_1 + a_2) \mid a_0, a_1, a_2 \in \mathbb{R} \}
$$

From our previous analysis, any vector $(x, y) \in \mathbb{R}^2$ can be written as:

$$
x = a_0
$$
$$
y = a_0 + a_1 + a_2
$$

Given x, for any y, choose:

$$
a_0 = x
$$

$$
a_1 + a_2 = y - x
$$

since a_1, a_2 are arbitrary subject only to their sum, the image is all of $\mathbb{R}^2$:

$$
\operatorname{Im}(T) = \mathbb{R}^2
$$

meaning T is surjective.

Summary of T's Key Properties

Property	Description
Linearity	T is linear.
Kernel	$\ker(T) = \text{span}\{ t - t^2 \}$ (dimension 1).
Image	\(\operatorname{Im}(T) = \mathbb{R

Frequently Asked Questions

What is the linear transformation T from P_2 to R defined by T(p) = [p(0), p(1)] for polynomial p(t)?

T is a linear transformation that maps a polynomial p(t) in P_2 to a vector in R, where the first component is p(0) and the second component is p(1).

How do you find T(p) for a specific polynomial p(t) in P_2?

To find T(p), evaluate p at t=0 and t=1, then form the vector [p(0), p(1)].

What is the matrix representation of T with respect to the standard basis of P_2?

The matrix representation is constructed by applying T to the basis polynomials 1, t, t^2 and expressing the results as vectors in R^2. For example, T(1) = [1, 1], T(t) = [0, 1], T(t^2) = [0, 1], leading to the matrix [[1, 0, 0], [1, 1, 1]].

Is the transformation T from P_2 to R injective? Why or why not?

No, T is not injective because different polynomials can map to the same vector in R, especially if the polynomials differ by a polynomial that vanishes at 0 and 1.

Is the transformation T from P_2 to R surjective? Why or why not?

Yes, T is surjective because for any vector [a, b] in R, there exists a polynomial p(t) in P_2 such that p(0) = a and p(1) = b, for example, p(t) = a + (b - a)t.

How can T be used to evaluate polynomials at specific points?

T directly evaluates the polynomial at t=0 and t=1, so it can be used to quickly obtain these values, which are useful in polynomial interpolation and analysis.

Additional Resources

Linear Transformation: Unlocking the Structure of Polynomial Mapping from P_2 to $\mathbb{R}$

Introduction: The Power and Precision of Linear Transformations

In the realm of linear algebra, the concept of a linear transformation serves as a fundamental bridge connecting abstract algebraic structures with concrete geometric and analytical interpretations. When these transformations involve polynomials, especially those from the space P_2 (the space of all polynomials of degree at most 2), they open a window into understanding how polynomial functions behave under various operations and mappings.

Today, we delve into a specific but intriguing linear transformation $T: P_2 \to \mathbb{R}$, defined by:

$$
T(p) = [p(0), p(1)]
$$

This transformation takes a polynomial $p(t) \in P_2$ and maps it to a 2-dimensional real vector comprising the evaluations of $p(t)$ at $t=0$ and $t=1$. Such a transformation is not only mathematically elegant but also practically significant, offering insights into polynomial behavior at specific points—an essential aspect in approximation theory, numerical analysis, and computational algebra.

In this article, we will explore the structure, properties, and implications of this linear transformation in depth, providing a comprehensive understanding that marries theory with application.

Understanding the Domain and Codomain: P_2 and $\mathbb{R}$

The Polynomial Space P_2

The space P_2 consists of all polynomials of degree at most 2, which can be written in the general form:

$$
p(t) = a t^2 + b t + c
$$

where $a, b, c \in \mathbb{R}$.

Basis of P_2: A standard basis for P_2 is:

$$
\{1, t, t^2\}
$$

Any polynomial $p(t)$ in P_2 can be expressed as a linear combination of these basis elements:

$$
p(t) = c \cdot 1 + b \cdot t + a \cdot t^2
$$

This basis is crucial because it allows us to analyze the linear transformation's action via matrix representations and to understand how it maps a general polynomial.

The Codomain $\mathbb{R}^2$

The target space $\mathbb{R}^2$ is a simple two-dimensional real vector space, consisting of all ordered pairs:

$$
\begin{bmatrix}
x \\
y
\end{bmatrix}
$$

In our case, the transformation maps each polynomial to a vector containing its evaluations at two specific points—0 and 1.

Defining the Transformation T

Given the polynomial $p(t)$, the transformation is explicitly:

$$
T(p) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}
$$

This seemingly straightforward mapping encapsulates some profound concepts:

- Evaluation points: The transformation focuses on the polynomial's values at two fixed points.
- Linearity: Since the evaluation at points is a linear operation, T preserves addition and scalar multiplication.

Let's analyze this transformation step-by-step.

Step 1: Expressing T in Terms of Polynomial Coefficients

Suppose $p(t) = a t^2 + b t + c$. Then:

$$
p(0) = c
$$
$$
p(1) = a \cdot 1^2 + b \cdot 1 + c = a + b + c
$$

Thus, the transformation can be written as:

$$
T(p) = \begin{bmatrix} c \\ a + b + c \end{bmatrix}
$$

or, equivalently,

$$
T(p) = \begin{bmatrix} c \\ a + b + c \end{bmatrix}
$$

This directly links the polynomial coefficients to the image under T.

Step 2: Matrix Representation of T

To understand the transformation's structure explicitly, express T as a matrix relative to the basis $\{1, t, t^2\}$.

Given $p(t) = a t^2 + b t + c$, the coordinate vector in P_2 is:

$$
\vec{p} = \begin{bmatrix} c \\ b \\ a \end{bmatrix}
$$

The evaluation at 0 is:

\[
p(0) = c
\]

which corresponds to the linear functional:

\[
L_0(p) = c
\]

Similarly, evaluation at 1:

\[
p(1) = a + b + c
\]

The combined action of \(T \) can be represented by the matrix:

\[
[T] =
\begin{bmatrix}
\text{coefficients for } p(0) \\
\text{coefficients for } p(1)
\end{bmatrix}
\]

Expressed as matrix multiplication:

\[
T(p) = \begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
c \\
b \\
a
\end{bmatrix}
\]

or, explicitly:

\[
T(p) =
\begin{bmatrix}
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}

c \\
b \\
a
\end{bmatrix}
\]

This matrix succinctly captures the linear transformation's action, mapping the coefficient vector to the evaluation vector.

Step 3: Analyzing the Properties of \(T \)

Understanding the transformation's properties is essential for grasping its behavior and applications.

Linearity

Because evaluations are linear functionals, \(T \) is linear:

\[
T(p + q) = T(p) + T(q)
\]
\[
T(\lambda p) = \lambda T(p)
\]

for all \(p, q \in P_2 \) and \(\lambda \in \mathbb{R} \).

Kernel of \(T \)

The kernel of \(T \) consists of all polynomials \(p(t) \) such that:

\[
T(p) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

which implies:

\[
p(0) = 0 \quad \text{and} \quad p(1) = 0
\]

The set of all polynomials vanishing at 0 and 1 is:

\[
\operatorname{ker}(T) = \{ p(t) \in P_2 : p(0) = 0, p(1) = 0 \}
\]

Since \(p(t) \) is quadratic at most, the general form of such polynomials is:

$$
p(t) = k t (t - 1)
$$

where $k \in \mathbb{R}$. Therefore, the kernel is a one-dimensional subspace spanned by:

$$
p(t) = t(t - 1)
$$

Range of T

The range of T is the set of all vectors in $\mathbb{R}^2$ that can be obtained from some polynomial in P_2.

Given the matrix representation, the range is:

$$
\operatorname{Range}(T) = \left\{ \begin{bmatrix} c \\ a + b + c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}
$$

Since a, b, c are arbitrary, the range covers all of $\mathbb{R}^2$. Therefore, T is surjective.

Step 4: Determining the Rank and Nullity

Using the rank-nullity theorem:

$$
\dim(P_2) = 3
$$

and

$$
\operatorname{rank}(T) = \dim(\operatorname{Range}(T))
$$

From the previous discussion, the rank of T is 2 (since the image is all of $\mathbb{R}^2$) and the nullity (dimension of the kernel) is 1:

$$
\operatorname{nullity}(T) = 3 - 2 = 1
$$

This aligns with the earlier conclusion that the kernel is spanned by $t(t-1)$.

Step 5: Practical Applications and Intuitive Insights

The structure of this transformation reveals several practical applications:

1. Polynomial Interpolation and Approximation:
- The evaluations at 0 and 1 serve as data points.
- The kernel's polynomials vanish at those points, representing the space of functions that "disappear" at the specified points.
- Such mappings underpin interpolation schemes where polynomial values are matched at specific data points.

2. Numerical Analysis:
- Evaluations at fixed points are fundamental in numerical methods, including finite element methods and polynomial collocation.
- Understanding the linear transformation helps optimize algorithms for polynomial approximation.

3. Control and Signal

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