Set Up The Integral For The Electric Field At A Distance Z From The Centre Of A Sphere Or Radius R Which

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Understanding how to determine the electric field at a specific point in space due to a charged object is a fundamental aspect of electrostatics. When dealing with symmetric charge distributions such as spheres, the problem simplifies significantly due to symmetry considerations. This article provides a comprehensive guide to setting up the integral for calculating the electric field at a distance Z from the center of a uniformly charged sphere of radius R. We will explore the problem step-by-step, covering the physical principles, the mathematical formulation, and the integral setup necessary for accurate computation.

Introduction to Electric Fields and Symmetric Charge Distributions

The electric field, denoted as E, is a vector field representing the force experienced by a unit positive charge placed at a point in space. When calculating E due to a charge distribution, Coulomb's law is the foundational principle:

$$ \mathbf{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathrm{d}q}{r^2} \hat{\mathbf{r}} $$

where:

- $\mathrm{d}q$ is an infinitesimal element of charge,
- r is the distance from the charge element to the point where the field is being calculated,
- $\hat{\mathbf{r}}$ is the unit vector pointing from the charge element to the point,
- ε_0 is the vacuum permittivity.

When the charge distribution is symmetric, such as a sphere with uniform charge density, the problem simplifies significantly because of symmetry considerations. Specifically, the electric field at a point outside or inside the sphere can be derived using the principle of superposition, considering contributions from all charge elements.

Physical Setup and Assumptions

To accurately set up the integral, understanding the physical configuration is essential. Consider the following scenario:

- A sphere of radius R carries a total charge Q, distributed uniformly throughout its volume.
- The charge density ρ is constant and given by:

$$ \rho = \frac{Q}{\frac{4}{3}\pi R^3} $$

- The goal is to find the electric field $\mathbf{E}$ at a point located a distance Z from the center of the sphere along a chosen axis (say, the z-axis).

The problem reduces to calculating the contribution of all infinitesimal charge elements within the sphere to the electric field at the specified point.

Coordinate System and Symmetry Considerations

Establish a coordinate system to facilitate the integral setup:

- Place the center of the sphere at the origin $(0, 0, 0)$.
- Align the z-axis with the line connecting the center to the point where the electric field is to be calculated.
- The point of observation P is located at $(0, 0, z)$.

Due to symmetry, the problem simplifies because:

- The contributions to the electric field from charge elements located symmetrically around the axis cancel in the perpendicular directions.
- Only the components along the z-axis (the axis of symmetry) contribute to the net electric field.

Mathematical Formulation of the Problem

To set up the integral:

1. Identify the Charge Elements

Consider an infinitesimal volume element $\mathrm{d}V$ within the sphere at position (r', θ', ϕ') in spherical coordinates:

$$
\mathrm{d}V = r'^2 \sin \theta' \, \mathrm{d}r' \, \mathrm{d}\theta' \, \mathrm{d}\phi'
$$

The charge of this element:

$$
\mathrm{d}q = \rho \, \mathrm{d}V
$$

2. Determine the Position of the Charge Element

- Coordinates of the charge element:

$$
\mathbf{r'} = r' \sin \theta' \cos \phi' \, \hat{\mathbf{x}} + r' \sin \theta' \sin \phi' \, \hat{\mathbf{y}} + r' \cos \theta' \, \hat{\mathbf{z}}
$$

- The observation point is at $\mathbf{R} = 0, 0, z$.

3. Calculate the Vector from the Charge Element to the Observation Point

$$
\mathbf{R} - \mathbf{r'} = (0 - r' \sin \theta' \cos \phi') \hat{\mathbf{x}} + (0 - r' \sin \theta' \sin \phi') \hat{\mathbf{y}} + (Z - r' \cos \theta') \hat{\mathbf{z}}
$$

The magnitude:

$$
|\mathbf{R} - \mathbf{r'}| = \sqrt{r'^2 + Z^2 - 2 r' Z \cos \theta'}
$$

Setting Up the Integral for the Electric Field

Based on the symmetry and Coulomb's law, the differential electric field contribution $\mathrm{d}\mathbf{E}$ from an element of charge $\mathrm{d}q$ is:

$$
\mathrm{d}\mathbf{E} = \frac{1}{4\pi \varepsilon_0} \frac{\mathrm{d}q}{|\mathbf{R} - \mathbf{r'}|^2} \hat{\mathbf{r}}
$$

where:

- $\hat{\mathbf{r}}$ is the unit vector pointing from the charge element to the observation point:

$$
\hat{\mathbf{r}} = \frac{\mathbf{R} - \mathbf{r'}}{|\mathbf{R} - \mathbf{r'}|}
$$

Given the symmetry, only the z-component survives after integration over ϕ', because perpendicular components cancel out.

The Integral Expression

The total electric field at Z along the z-axis:

$$
E_z(Z) = \frac{1}{4 \pi \varepsilon_0} \int \frac{\mathrm{d}q}{|\mathbf{R} - \mathbf{r'}|^2} \cos \theta'
$$

which can be expressed explicitly as:

$$
E_z(Z) = \frac{\rho}{4 \pi \varepsilon_0} \int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{R} \frac{r'^2 \sin \theta' \, \mathrm{d}r' \, \mathrm{d}\theta' \, \mathrm{d}\phi'}{\left(r'^2 + Z^2 - 2 r' Z \cos \theta' \right)} \times \frac{Z - r' \cos \theta'}{\sqrt{r'^2 + Z^2 - 2 r' Z \cos \theta'}}
$$

The integration over ϕ' yields a factor of 2π because the integrand does not depend on ϕ'.

Simplification Using Symmetry and Integral Reduction

The integral reduces to:

$$
E_z(Z) = \frac{\rho}{2 \varepsilon_0} \int_{0}^{R} r'^2 \, \mathrm{d}r' \int_{0}^{\pi} \frac{\sin \theta' (Z - r' \cos \theta')}{\left(r'^2 + Z^2 - 2 r' Z \cos \theta' \right)^{3/2}} \, \mathrm{d}\theta'
$$

This form is suitable for further analytical or numerical evaluation.

Special Cases and Evaluation

- Outside the Sphere (Z > R):

When the observation point is outside the sphere, the integral simplifies to the electric field of a point charge:

$$
E_z(Z) = \frac{Q}{4\pi \varepsilon_0 Z^2}
$$

- Inside the Sphere (Z < R):

When the point is inside the sphere, the integral must be evaluated carefully, often leading to the known result:

$$
E_z(Z) = \frac{Q Z}{4 \pi \varepsilon_0 R^3}
$$

which is derived using Gauss's law.

Practical Tips for Setting Up the Integral

To effectively set up the integral for the electric field at a distance Z:

1. Choose a Coordinate System Aligned with the Observation Point

- Align the z-axis to pass through the charge distribution and the point where the field is measured.

2. Identify Symmetry to Simplify the Problem

- Use spherical coordinates for a sphere.
- Recognize that components perpendicular to the axis cancel due to symmetry.

3. Express the Differential Charge Element

- Use $\mathrm{d}q = \rho \, \mathrm{d}V$.

4. Write the Distance and Direction Vectors

-

Frequently Asked Questions

How do I set up the integral for the electric field at a point located a distance Z from the center of a uniformly charged sphere of radius R?

To set up the integral, consider the sphere's surface as composed of infinitesimal charge elements. Use spherical coordinates with the z-axis passing through the observation point. Express the charge element dq in terms of surface charge density σ and surface area element dA. The electric field contribution dE from each element is given by Coulomb's law, integrated over the sphere's surface, considering the vector distance from each charge element to the point at distance Z.

What is the expression for the differential charge element on the sphere's surface in this setup?

The differential charge element is $dq = \sigma \, dA$, where σ is the surface charge density, and $dA = R^2 \sin\theta \, d\theta \, d\phi$ for a spherical surface. Here, θ is the polar angle measured from the z-axis, and φ is the azimuthal angle.

How do I account for the geometry when integrating to find the electric field at a point outside the sphere ($Z > R$)?

When $Z > R$, the symmetry simplifies the problem. The electric field is directed along the line connecting the sphere's center and the point. Use the law of cosines to relate the distances and set up the integral over the sphere's surface, integrating the contributions from all charge elements, considering the angle between the position vector of each element and the line to the point.

What simplifications are possible when the point is outside versus inside the sphere?

Outside the sphere ($Z > R$), the sphere can be treated as a point charge with total charge Q, simplifying the integral to Coulomb's law: $E = (1/4\pi\varepsilon_0)\, Q / Z^2$. Inside the sphere ($Z < R$), the integral accounts for only the charge enclosed within radius Z, leading to a different expression where the electric field varies linearly with Z.

How does symmetry help in reducing the complexity of setting up the integral for the electric field?

Symmetry allows us to eliminate components of the electric field that cancel out due to the uniform charge distribution. For a sphere, the contributions in directions perpendicular to the axis of observation cancel, leaving only the component along the axis. This reduces the problem to a single integral over the relevant angular variables, often simplifying the calculation.

Additional Resources

Set Up The Integral For The Electric Field At A Distance Z From The Centre Of A Sphere Of Radius R Which

Understanding the electric field created by charged objects is fundamental in electrostatics, a core branch of physics. When dealing with symmetrical charge distributions such as spheres, it's critical to develop a systematic approach to computing the electric field at a specific point in space. In this discussion, we explore how to set up the integral for the electric field at a distance \(Z \) from the center of a uniformly charged sphere of radius R. This process involves carefully considering the geometry, symmetry, and the principles of Coulomb's law, leading to an integral formulation suitable for analytical evaluation.

Understanding the Physical Context and Assumptions

Before delving into the mathematical formulation, it is essential to clarify the scenario and assumptions involved:

- Uniform Charge Distribution: The sphere has a total charge Q uniformly spread throughout its volume, leading to a constant volumetric charge density:

$$
\rho = \frac{Q}{\frac{4}{3}\pi R^3}
$$

- Observation Point: The point where the electric field is to be calculated is located along the axis passing through the center of the sphere, at a distance Z from the center. This simplifies the symmetry considerations, as the problem reduces to a one-dimensional analysis along the axis.

- Coordinate System Choice: It's most convenient to place the sphere at the origin of a coordinate system, with the z-axis aligned along the line connecting the sphere's center to the observation point. The observation point is at $z = Z$.

- Electrostatics Principles: Coulomb's law governs the electric field contribution from each charge element. The principle of superposition allows summing the contributions from all infinitesimal charge elements within the sphere.

Mathematical Foundations for Setting Up the Integral

The goal is to express the electric field $\mathbf{E}$ at the point P (located at Z along the z-axis) as an integral over the volume of the sphere:

$$
\mathbf{E}(Z) = \int \mathrm{d}\mathbf{E}
$$

where each $\mathrm{d}\mathbf{E}$ is the contribution from an infinitesimal charge element $\mathrm{d}q$. The steps involve:

1. Defining the charge element $\mathrm{d}q$: Based on the uniform charge density.
2. Determining the position of $\mathrm{d}q$: Using spherical coordinates.
3. Expressing Coulomb's law: To find the contribution $\mathrm{d}\mathbf{E}$ at the observation point.
4. Utilizing symmetry: To simplify the vector sum, especially along the axis.

Coordinate System and Charge Element Placement

Using spherical coordinates centered at the sphere's center:

- Coordinates of a charge element $\mathrm{d}q$:

$$
(r', \theta', \phi')
$$

where:
- r' is the radial distance from the sphere's center ($0 \leq r' \leq R$)
- θ' is the polar angle measured from the z-axis ($0 \leq \theta' \leq \pi$)
- ϕ' is the azimuthal angle in the xy-plane ($0 \leq \phi' \leq 2\pi$)

- Position of the charge element:

$$
\mathbf{r'} = r' \sin \theta' \cos \phi' \, \hat{\mathbf{x}} + r' \sin \theta' \sin \phi' \, \hat{\mathbf{y}} + r' \cos \theta' \, \hat{\mathbf{z}}
$$

- Position of the observation point:

$$
\mathbf{R} = Z \hat{\mathbf{z}}
$$

Charge Density and Differential Charge Element

Given the uniform volumetric charge density ρ:

$$
\rho = \frac{Q}{\frac{4}{3}\pi R^3}
$$

The differential volume element in spherical coordinates:

$$
\mathrm{d}V = r'^2 \sin \theta' \, \mathrm{d} r' \, \mathrm{d} \theta' \, \mathrm{d} \phi'
$$

Hence, the differential charge:

$$
\mathrm{d}q = \rho \, \mathrm{d}V = \rho r'^2 \sin \theta' \, \mathrm{d} r' \, \mathrm{d} \theta' \, \mathrm{d} \phi'
$$

Expressing Coulomb's Law for the Differential Element

The electric field contribution from $\mathrm{d}q$ at the observation point is:

$$
\mathrm{d}\mathbf{E} = \frac{1}{4 \pi \varepsilon_0} \frac{\mathrm{d}q}{|\mathbf{R} - \mathbf{r'}|^2} \hat{\mathbf{u}}
$$

where:

- $|\mathbf{R} - \mathbf{r'}|$ is the distance between the charge element and the observation point:

$$
|\mathbf{R} - \mathbf{r'}| = \sqrt{(Z - r' \cos \theta')^2 + (r' \sin \theta')^2} = \sqrt{r'^2 - 2 r' Z \cos \theta' + Z^2}
$$

- $\hat{\mathbf{u}}$ is the unit vector pointing from the charge element to the observation point:

$$
\hat{\mathbf{u}} = \frac{\mathbf{R} - \mathbf{r'} }{ |\mathbf{R} - \mathbf{r'}| }
$$

Since the problem involves symmetry about the z-axis, the transverse components (perpendicular to the axis) cancel out when integrating over ϕ'. Only the axial component (along z) survives, simplifying the integral considerably.

Evaluating the Electric Field Components

Due to symmetry:

- The x and y components of $\mathrm{d}\mathbf{E}$ integrate to zero because for each charge element at ϕ', there's a corresponding element at $\phi' + \pi$ with equal and opposite transverse field components.

- The only non-zero component of the electric field is along the z-axis:

$$
E_z(Z) = \int \mathrm{d}E_z
$$

Expressed as:

$$
\mathrm{d}E_z = \frac{1}{4\pi \varepsilon_0} \frac{\mathrm{d}q}{|\mathbf{R} - \mathbf{r'}|^2} \cos \alpha
$$

where α is the angle between $\hat{\mathbf{u}}$ and the z-axis:

$$
\cos \alpha = \frac{Z - r' \cos \theta'}{|\mathbf{R} - \mathbf{r'}|}
$$

Putting it all together:

$$
\mathrm{d}E_z = \frac{1}{4 \pi \varepsilon_0} \frac{\mathrm{d}q}{|\mathbf{R} - \mathbf{r'}|^2} \frac{Z - r' \cos \theta'}{|\mathbf{R} - \mathbf{r'}|}
$$

which simplifies to:

$$
\mathrm{d}E_z = \frac{1}{4 \pi \varepsilon_0} \frac{\mathrm{d}q (Z - r' \cos \theta')}{|\mathbf{R} - \mathbf{r'}|^3}
$$

Formulating the Integral for the Electric Field

Substituting the expressions for $\mathrm{d}q$ and $|\mathbf{R} - \mathbf{r'}|$:

$$
E_z(Z) = \frac{1}{4 \pi \varepsilon_0} \int_{r'=0}^{R} \int_{\theta'=0}^{\pi} \int_{\phi'=0}^{2\pi} \frac{\rho r'^2 \sin \theta' \, \mathrm{d} r' \, \mathrm{d} \theta' \, \mathrm{d} \phi' \cdot (Z - r' \cos \theta')}{\left(r'^2 - 2 r' Z \cos \theta' + Z^2 \right)^{3/2}}
$$

Since the integrand does not depend on ϕ', integration over ϕ' yields a factor of 2π, leading to:

\[
E_z(Z) = \frac{\rho}{2 \varepsilon

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