Solve Each System By Using Elementary Row Operations On The Equations Or On The Augmented Matrix. Follow

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Solving systems of linear equations is a fundamental aspect of algebra and linear algebra, with applications spanning engineering, computer science, economics, and more. One of the most systematic and efficient methods for solving such systems is through elementary row operations applied to the augmented matrix representation of the system. This approach not only simplifies the process but also provides a clear pathway to understanding the structure of the solutions, whether they are unique, infinite, or nonexistent. In this article, we will explore the principles, procedures, and best practices for solving systems by utilizing elementary row operations, guiding you step-by-step through the process and illustrating key concepts with examples.

Understanding Systems of Linear Equations

What Is a System of Linear Equations?

A system of linear equations consists of multiple equations involving the same set of variables. For example:

$$
\begin{cases}
ax + by + cz = d \\
ex + fy + gz = h \\
ix + jy + kz = l
\end{cases}
$$

Here, each equation is linear, and the goal is to find the values of x, y, z that satisfy all equations simultaneously.

Types of Solutions

Depending on the system's structure, solutions can be classified as:
- Unique solution: Exactly one solution exists.
- Infinite solutions: The system has infinitely many solutions.
- No solution: The system is inconsistent.

Representing Systems Using Augmented Matrices

What Is an Augmented Matrix?

An augmented matrix is a compact way to represent a system of linear equations. It combines the coefficients of the variables and the constants from the right-hand side into a single matrix:

$$
\left[\begin{array}{ccc|c}
a_{11} & a_{12} & a_{13} & d_1 \\
a_{21} & a_{22} & a_{23} & d_2 \\
a_{31} & a_{32} & a_{33} & d_3
\end{array}\right]
$$

This matrix form allows for systematic manipulation via elementary row operations to find the solution set.

Advantages of Using Augmented Matrices

- Simplifies complex systems into a standard form.
- Facilitates the use of matrix operations for solution.
- Provides a visual and procedural approach to solving.

Elementary Row Operations

Definition and Purpose

Elementary row operations are the basic manipulations that can be performed on the rows of a matrix without changing its solution set. They are fundamental to the Gauss-Jordan elimination process.

Types of Elementary Row Operations

1. Row swapping (interchange): Swapping two rows.
- Example: $R_i \leftrightarrow R_j$
2. Row scaling (multiplication): Multiplying a row by a non-zero scalar.
- Example: $R_i \rightarrow k R_i$, where $k \neq 0$
3. Row addition (replacement): Adding a multiple of one row to another row.
- Example: $R_j \rightarrow R_j + k R_i$

Key Properties

- These operations do not change the solution set.

- They are used to convert the matrix into a form where solutions are easily identified.

Step-by-Step Process for Solving Systems

1. Write the Augmented Matrix

Begin by translating the system of equations into an augmented matrix.

2. Use Elementary Row Operations to Reach Row Echelon Form

Transform the matrix so that it has zeros below the leading entries (pivots) in each column. This involves:
- Creating a leading 1 in the first row, first column.
- Using row operations to eliminate entries below this pivot.
- Moving to the second row and second column, repeating the process.
- Continuing until the matrix is in upper triangular form.

3. Back Substitution (or Reduced Row Echelon Form)

Once in row echelon form, solve for the variables starting from the bottom row upward:
- If a row corresponds to an equation like $0 = c$ (where $c \neq 0$), the system is inconsistent (no solution).
- If the system has free variables, express some variables in terms of parameters to represent infinite solutions.
- If each variable is determined uniquely, the solution is the specific set of values obtained through back substitution.

4. Convert Back to Equations (if necessary)

Translate the solutions from the matrix form back into variable expressions.

Illustrative Example

System of Equations

\[
\begin{cases}

x + 2y + z = 9 \\
2x + 3y + 3z = 20 \\
-x + y + 2z = 5
\end{cases}
\]

Step 1: Write the Augmented Matrix

\[
\left[\begin{array}{ccc|c}
1 & 2 & 1 & 9 \\
2 & 3 & 3 & 20 \\
-1 & 1 & 2 & 5
\end{array}\right]
\]

Step 2: Row Operations to Achieve Row Echelon Form

- Keep \(R_1 \) as is.
- Eliminate the \(2\) in \(R_2 \) using \(R_1 \):
\[
R_2 \rightarrow R_2 - 2 R_1
\]
\[
R_2 = [0, -1, 1, 2]
\]
- Eliminate \(-1\) in \(R_3 \) using \(R_1 \):
\[
R_3 \rightarrow R_3 + R_1
\]
\[
R_3 = [0, 3, 3, 14]
\]

Updated matrix:
\[
\left[\begin{array}{ccc|c}
1 & 2 & 1 & 9 \\
0 & -1 & 1 & 2 \\
0 & 3 & 3 & 14
\end{array}\right]
\]

- Make the pivot in \(R_2 \) a 1 by multiplying by \(-1\):
\[
R_2 \rightarrow - R_2
\]
\[
R_2 = [0, 1, -1, -2]

$$
$$

- Use R_2 to eliminate the 3 in R_3:

$$
R_3 \rightarrow R_3 - 3 R_2
$$

$$
R_3 = [0, 0, 6, 20]
$$

- Divide R_3 by 6 to make the leading coefficient 1:

$$
R_3 \rightarrow \frac{1}{6} R_3
$$

$$
R_3 = [0, 0, 1, \frac{10}{3}]
$$

- Use R_3 to eliminate the z-term in R_1 and R_2:
- For R_1:

$$
R_1 \rightarrow R_1 - R_3
$$

$$
R_1 = [1, 2, 0, \frac{17}{3}]
$$

- For R_2:

$$
R_2 \rightarrow R_2 + R_3
$$

$$
R_2 = [0, 1, 0, \frac{1}{3}]
$$

Step 3: Back Substitution

- From R_3: $z = \frac{10}{3}$
- From R_2: $y + 0 = \frac{1}{3} \Rightarrow y = \frac{1}{3}$
- From R_1: $x + 2 y = \frac{17}{3} \Rightarrow x + 2 \times \frac{1}{3} = \frac{17}{3} \Rightarrow x + \frac{2}{3} = \frac{17}{3} \Rightarrow x = \frac{15}{3} = 5$

Solution:

$$
x = 5, \quad y = \frac{1}{3}, \quad z = \frac{10}{3}
$$

Special Cases and Considerations

Inconsistent Systems

If during the process, a row reduces to a form like:
\[
[0 \quad 0 \quad 0 \quad | \quad c], \quad c \neq 0
\]
then the system has no solution.

Dependent Systems