Solve The Given Differential Equation By Separation Of Variables.y Ln X * (dx/dy) = [(y+1)/x]^2[(y^2)/2]

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Introduction to Solving Differential Equations by Separation of Variables

Differential equations are fundamental in describing numerous phenomena in physics, engineering, biology, and other sciences. Among various methods for solving differential equations, the separation of variables stands out as a straightforward and powerful technique, especially for equations where variables can be separated into individual functions of x and y.

In this article, we will explore how to solve the differential equation:

y ln x (dx/dy) = [(y+1)/x]^2 (y^2/2)

using the method of separation of variables. We will break down each step, clarify the process, and provide insights into the underlying principles. By the end of this guide, you'll have a clear understanding of how to handle similar differential equations using this approach.

Understanding the Given Differential Equation

Let's analyze the equation:

y ln x (dx/dy) = [(y+1)/x]^2 (y^2/2)

This is a first-order differential equation involving both x and y, with the derivative dx/dy. Our goal is to manipulate the equation into a form where all x terms are on one side and all y terms on the other, enabling us to integrate both sides separately.

Key observations:

- The presence of ln x and x in the equation suggests potential substitution or rearrangement.
- The right side involves both y and x in a rational expression.
- The structure hints at the possibility of separating variables after algebraic manipulation.

Before proceeding, let's rewrite the equation for clarity and identify the method of approach.

Rearranging the Differential Equation

The original equation:

$$y \ln x \;\; (dx/dy) = [(y+1)/x]^2 \;\; (y^2/2)$$

Our first step is to isolate dx/dy:

$$dx/dy = \{ \; [(y+1)/x]^2 \;\; (y^2/2) \; \} / (y \ln x)$$

Expressed more explicitly:

$$dx/dy = [\; (y+1)^2 / x^2 \;] \;\; (\; y^2 / 2 \;) / (\; y \ln x \;)$$

Simplify numerator and denominator:

$$dx/dy = [\; (y+1)^2 \;\; y^2 / (2 \; x^2) \;] / (\; y \ln x \;)$$

Now, divide numerator by denominator:

$$dx/dy = [\; (y+1)^2 \;\; y^2 \;] / [\; 2 \; x^2 \;\; y \ln x \;]$$

Notice that y in numerator and denominator cancels:

$$dx/dy = [\; (y+1)^2 \;\; y \;] / [\; 2 \; x^2 \;\; \ln x \;]$$

This simplifies the differential equation to:

$$dx/dy = [\; y \; (y+1)^2 \;] / [\; 2 \; x^2 \ln x \;]$$

Preparing for Separation of Variables

Our goal is to write the differential equation in the form:

```
f(x) dx = g(y) dy
```

or equivalently, separate the x and y terms on different sides.

From the current form:

```
dx/dy = [ y (y+1)^2 ] / [ 2 x^2 ln x ]
```

Invert both sides to express dy/dx:

```
dy/dx = [ 2 x^2 ln x ] / [ y (y+1)^2 ]
```

Now, the variables are separated—x terms on one side, y terms on the other.

Separating Variables and Setting Up the Integral

Rewrite the differential equation:

```
dy/dx = [ 2 x^2 ln x ] / [ y (y+1)^2 ]
```

Cross-multiplied, it becomes:

```
y (y+1)^2 dy = 2 x^2 ln x dx
```

This is a separable differential equation, which we can integrate on both sides:

```
∫ y (y+1)^2 dy = ∫ 2 x^2 ln x dx
```

Now, our task is to compute these integrals separately.

Integrating the y-Related Side

Let's focus on the integral:

$$I_y = \int y (y+1)^2 dy$$

Expand $(y+1)^2$:

$$(y+1)^2 = y^2 + 2 y + 1$$

So,

$$I_y = \int y (y^2 + 2 y + 1) dy$$

Distribute y:

$$I_y = \int (y^3 + 2 y^2 + y) dy$$

Integrate term-by-term:

- $\int y^3 dy = y^4 / 4$
- $\int 2 y^2 dy = 2 y^3 / 3$
- $\int y dy = y^2 / 2$

Putting it all together:

$$I_y = y^4 / 4 + (2 y^3) / 3 + y^2 / 2 + C_y$$

Integrating the x-Related Side

Next, consider:

$$I_x = \int 2 x^2 \ln x dx$$

This integral involves $x^2 \ln x$, which suggests integration by parts.

Applying Integration by Parts:

Let:

- $u = \ln x \Rightarrow du/dx = 1/x$
- $dv = 2 x^2 dx \Rightarrow v = (2/3) x^3$

Using integration by parts:

$$I_x = uv - \int v \, du$$

Substitute:

$$I_x = \ln x \;\; (2/3) x^3 - \int (2/3) x^3 \;\; (1/x) \, dx$$

Simplify the integral:

$$I_x = (2/3) x^3 \ln x - (2/3) \int x^2 \, dx$$

Integrate x^2:

$$\int x^2 \, dx = x^3 / 3$$

Thus,

$$I_x = (2/3) x^3 \ln x - (2/3) \;\; (x^3 / 3) = (2/3) x^3 \ln x - (2/9) x^3 + C_x$$

Combining the Results and General Solution

Recall the separated integral equation:

$$I_y = I_x + C$$

substituting the computed integrals:

$$y^4 / 4 + (2 y^3) / 3 + y^2 / 2 = (2/3) x^3 \ln x - (2/9) x^3 + C$$

We can write the general solution compactly as:

$$\frac{y^4}{4} + \frac{2 y^3}{3} + \frac{y^2}{2} = \frac{2}{3} x^3 \ln x - \frac{2}{9} x^3 + C$$

where C is an arbitrary constant.

Summary of the Solution Steps

To summarize, the process involved:

1. Rearranging the original differential equation to express dy/dx.
2. Separating variables to facilitate integration.
3. Integrating both sides:
- The y-side integral using polynomial expansion.
- The x-side integral using integration by parts.
4. Combining the integrals to obtain the implicit solution.

Conclusion and Final Remarks

The method of separation of variables proves effective in solving the given differential equation, provided the equation can be manipulated algebraically to isolate variables. The key steps include algebraic rearrangement, integral computation, and proper application of integration techniques like integration by parts.

This approach is widely applicable to many first-order differential equations, especially those involving products or quotients of functions that can be separated neatly. Mastery of algebraic manipulation and integral calculus is essential for effectively employing this method.

In practice, solutions may sometimes be expressed implicitly, as in this case, or explicitly if possible. Always consider the domain restrictions imposed by logarithmic and polynomial expressions when interpreting the solution.

By following these systematic steps, you can confidently tackle similar differential equations and expand your problem-solving toolkit in differential equations.

Additional Resources for Further Study

- "Differential Equations and Boundary Value Problems" by Dennis G. Zill

- "Elementary Differential Equations and Boundary Value Problems" by William E. Boyce and Richard C. DiPrima
- Online tutorials on integration by parts and separation of variables techniques

Feel free to revisit each step for practice, and explore more complex equations to strengthen your understanding of solving differential equations via separation of variables.

Frequently Asked Questions

What is the first step to solve the differential equation $y \ln x \, (dx/dy) = [(y+1)/x]^2 (y^2/2)$ using separation of variables?

The first step is to rewrite the differential equation to isolate the variables x and y on different sides, typically by simplifying and rearranging terms to express (dx/dy) in terms of y, allowing separation of variables.

How can we simplify the given differential equation $y \ln x \, (dx/dy) = [(y+1)/x]^2 (y^2/2)$ for separation of variables?

Divide both sides of the equation by $y \ln x$ to isolate (dx/dy), resulting in $(dx/dy) = [((y+1)/x)^2 (y^2/2)] / (y \ln x)$. Then, express the right side to separate the x and y variables.

What substitution can be helpful in solving this differential equation by separation of variables?

A substitution such as $t = \ln x$ or $u = y$ could be helpful to simplify the equation, especially since $\ln x$ appears, making the variables more manageable for separation.

After separating variables, what integral forms will we need to evaluate to find the solution?

We will need to evaluate integrals of the form $\int dx / x$ and $\int$ functions of y dy, depending on how the variables are separated, to obtain the explicit solution.

What are some common challenges faced when solving this differential equation by separation of variables?

Challenges include correctly manipulating the equation to separate variables, handling complex algebraic expressions, and ensuring the integrals are properly evaluated, especially when involving logarithmic or polynomial terms.

Additional Resources

Comprehensive Guide to Solving the Differential Equation Using Separation of Variables

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Introduction

Differential equations are fundamental to understanding diverse phenomena in physics, engineering, biology, and many other sciences. Among various methods to solve these equations, separation of variables stands out as a straightforward and powerful technique, especially for equations where variables can be isolated on different sides of the equation.

In this detailed review, we will explore how to solve the specific differential equation:

$$
\boxed{
\ln x \cdot \frac{dy}{dx} = \left[\frac{y+1}{x}\right]^2 \cdot \frac{y^2}{2}
}
$$

using the separation of variables method. The goal is to understand each step thoroughly, analyze the underlying structure of the differential equation, and derive its solution systematically.

Understanding the Differential Equation

1. Recognizing the Form

The given differential equation is:

$$
\ln x \cdot \frac{dy}{dx} = \left(\frac{y+1}{x}\right)^2 \cdot \frac{y^2}{2}
$$

At first glance, the equation involves:

- A product of $\ln x$ with the derivative $\frac{dy}{dx}$,
- The right side contains a rational expression involving y and x,
- A quadratic term in $y+1$,
- The presence of both x and y in a multiplicative and divisive manner.

This mixture suggests that the variables are intertwined but potentially separable after appropriate algebraic manipulation.

Step 1: Simplify and Rearrange the Equation

Before applying separation of variables, it is helpful to rewrite the equation in a more manageable form.

$$
\ln x \cdot \frac{dy}{dx} = \frac{(y+1)^2}{x^2} \cdot \frac{y^2}{2}
$$

Divide both sides by $\ln x$ (assuming $x > 0$, since $\ln x$ must be defined):

$$
\frac{dy}{dx} = \frac{(y+1)^2}{x^2} \cdot \frac{y^2}{2 \ln x}
$$

Thus, the differential equation becomes:

$$
\frac{dy}{dx} = \frac{(y+1)^2 y^2}{2 x^2 \ln x}
$$

This form hints at potential substitution strategies, but our goal remains to separate the variables x and y.

Step 2: Isolating $\frac{dy}{dx}$

Rewrite as:

$$
dy = \frac{(y+1)^2 y^2}{2 x^2 \ln x} \, dx
$$

Or equivalently:

$$
dy = \left[\frac{(y+1)^2 y^2}{2}\right] \cdot \frac{1}{x^2 \ln x} \, dx
$$

The structure suggests that if we can express the right side as a product of a function of y and a function of x, then separation is straightforward.

Step 3: Variable Substitution Strategy

Given the complex dependence, consider potential substitutions:

- Substitution for y: The terms involve y and $y+1$, perhaps substitution $z = y + 1$.

- Substitution for x: There is a $\ln x$ term, suggesting that setting $t = \ln x$ might simplify the x dependence.

Let's analyze both options.

Step 4: Substitution $t = \ln x$

This is a common substitution when dealing with $\ln x$:

$$
t = \ln x \quad \Rightarrow \quad x = e^t
$$

Differentiating:

$$
\frac{dx}{dt} = e^t = x
$$

Express dy/dx in terms of dy/dt:

$$
\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{1}{x}
$$

Rearranged:

$$
dy/dt = x \cdot \frac{dy}{dx}
$$

Recall from earlier:

$$
\frac{dy}{dx} = \frac{(y+1)^2 y^2}{2 x^2 t}
$$

Substitute into the expression:

$$
dy/dt = x \times \frac{(y+1)^2 y^2}{2 x^2 t} = \frac{(y+1)^2 y^2}{2 x t}
$$

But since $x = e^t$, this simplifies to:

$$
dy/dt = \frac{(y+1)^2 y^2}{2 e^t t}
$$

Now, the differential equation in terms of t and y is:

$$

\boxed{
\frac{dy}{dt} = \frac{(y+1)^2 y^2}{2 e^t t}
}
\]

This is a more manageable form, as the variables are separated into functions of \(y\) and \(t\).

Step 5: Expressing as a Separable Equation in \(y\) and \(t\)

Rearranged:

\[
dy = \frac{(y+1)^2 y^2}{2 e^t t} dt
\]

Note that the right side is a product of functions in \(y\) and \(t\). To facilitate separation, rewrite as:

\[
\frac{dy}{(y+1)^2 y^2} = \frac{1}{2 t e^{t}} dt
\]

Now, the equation is explicitly separable:

\[
\boxed{
\frac{dy}{(y+1)^2 y^2} = \frac{1}{2 t e^{t}} dt
}
\]

Step 6: Partial Fraction Decomposition in \(y\)

The integral on the left involves:

\[
\int \frac{dy}{(y+1)^2 y^2}
\]

To integrate this, perform partial fractions:

\[
\frac{1}{(y+1)^2 y^2} = \frac{A}{y} + \frac{B}{y^2} + \frac{C}{y+1} + \frac{D}{(y+1)^2}
\]

Multiply both sides by \(y^2 (y+1)^2\):

\[
1 = A y (y+1)^2 + B (y+1)^2 + C y^2 (y+1) + D y^2

\]

Expand each term:

- \(A y (y+1)^2 = A y (y^2 + 2 y + 1) = A y^3 + 2A y^2 + A y\)

- \(B (y+1)^2 = B (y^2 + 2 y + 1) = B y^2 + 2 B y + B\)

- \(C y^2 (y+1) = C y^2 y + C y^2 = C y^3 + C y^2\)

- \(D y^2 = D y^2\)

Summing all:

\[
1 = (A y^3 + C y^3) + (2A y^2 + B y^2 + C y^2 + D y^2) + (A y + 2 B y) + B
\]

Group similar powers:

- \(y^3\): \((A + C) y^3\)

- \(y^2\): \((2A + B + C + D) y^2\)

- \(y\): \((A + 2 B) y\)

- Constant term: \(B\)

Since the right side is 1 (a constant), the coefficients of powers of \(y\) must be zero, and the constant term must be 1:

\[
\begin{cases}
A + C = 0 \quad &\text{(coefficient of } y^3) \\
2A + B + C + D = 0 \quad &\text{(coefficient of } y^2) \\
A + 2 B = 0 \quad &\text{(coefficient of } y) \\
B = 1 \quad &\text{(constant term)}
\end{cases}
\]

From \(B=1\), substitute into the equations:

- \(A + 2(1) = 0 \Rightarrow A = -2\)

- \(A + C = 0 \Rightarrow -2 + C = 0 \Rightarrow C=2\)

- \(2A + B + C + D= 0 \Rightarrow 2(-2) + 1 + 2 + D=0 \Rightarrow -4 + 3 + D=0 \Rightarrow D=1\)

Thus, the partial fraction decomposition is:

\[
\frac{1}{(y

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