(1 Point) Let F = 5xi + 5yj And Let N Be The Outward Unit Normal Vector To The Positively Oriented Circle

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Introduction to Vector Fields and Normal Vectors

Understanding vector fields and their associated normal vectors is fundamental in multivariable calculus, especially when dealing with line integrals, flux, and Green’s theorem. In this context, we analyze a specific vector field and the properties of the outward unit normal vector to a positively oriented circle. This exploration provides insights into how vector fields interact with curves and how to compute fluxes across these boundaries.

Defining the Vector Field F

Component-wise Description of F

The vector field F is given by:

$$ F = 5x\, \mathbf{i} + 5y\, \mathbf{j} $$

This notation indicates that:

- The x-component of F is $5x$
- The y-component of F is $5y$

Graphically, at any point (x, y), the vector F points outward from the origin, scaled proportionally to the coordinates.

Properties of F

- Radial Nature: Since both components are proportional to x and y, the vector field radiates outward from the origin.
- Divergence: The divergence of F is:

$$
\nabla \cdot F = \frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(5y) = 5 + 5 = 10
$$

- Curl: The curl in two dimensions (scalar form):

$$
\nabla \times F = \frac{\partial (5y)}{\partial x} - \frac{\partial (5x)}{\partial y} = 0 - 0 = 0
$$

This indicates that F is irrotational, i.e., curl-free, which has implications for the flux and circulation calculations.

The Positively Oriented Circle and Its Normal Vector

Definition of the Circle

Suppose we consider a circle C in the plane with:

- Center at the origin: $(0, 0)$
- Radius r, which is a positive real number

The circle's equation is:

$$
x^2 + y^2 = r^2
$$

Positively Oriented Circle

- Orientation: Counterclockwise direction

- Significance: Positive orientation is essential in applying the Green’s theorem and calculating line integrals, flux, or circulation around the boundary.

Outward Unit Normal Vector \(N \)

The outward unit normal vector \(N \) at any point on the circle points directly away from the center. For a circle centered at the origin:

\[
N = \frac{\mathbf{r}}{r} = \frac{x\, \mathbf{i} + y\, \mathbf{j}}{r}
\]

where \(\mathbf{r} = x\, \mathbf{i} + y\, \mathbf{j} \) is the position vector from the origin to a point on the circle.

Key points:

- The vector \(N \) is unit length: \(|N| = 1 \).
- It is outward pointing, indicating the direction normal to the boundary, away from the interior.

Mathematical Formulation of the Normal Vector

Explicit Expression of \(N \)

Given any point \((x, y) \) on the circle:

\[
N = \frac{x\, \mathbf{i} + y\, \mathbf{j}}{r}
\]

This vector is perpendicular to the tangent at that point and points outward.

Normal Vector in Terms of Circle Parameterization

Suppose the circle is parameterized as:

\[

x(t) = r \cos t, \quad y(t) = r \sin t, \quad t \in [0, 2\pi]
\]

Then:

\[
N(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j}
\]

which confirms that \(N \) is the unit vector pointing radially outward.

Applications of Normal Vectors and Vector Fields in Calculus

Calculating Flux Across the Circle

The flux of \(F \) across the circle \(C \) in the outward normal direction is given by the line integral:

\[
\Phi = \oint_C F \cdot N\, ds
\]

where:

- \(ds \) is the differential arc length
- \(N \) is the outward unit normal vector

Applying this integral involves:

1. Parameterizing the circle
2. Computing the dot product \(F \cdot N \)
3. Integrating over the parameter domain

Using Green’s Theorem

Green’s theorem relates the line integral around a simple closed curve to a double integral over the region it encloses:

\[

\oint_C (P\, dx + Q\, dy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA
\]

where \(D \) is the disk bounded by \(C \).

In our case:

- \(P = 5x \)
- \(Q = 5y \)

The divergence (which is relevant for flux calculations) is:

\[
\nabla \cdot F = 10
\]

Thus, the flux through the disk:

\[
\text{Flux} = \iint_D (\nabla \cdot F) dA = 10 \times \text{Area of } D = 10 \pi r^2
\]

Step-by-Step Calculation of the Flux

Parameterization of the Circle

Using:

\[
x(t) = r \cos t, \quad y(t) = r \sin t, \quad t \in [0, 2\pi]
\]

The differential arc length:

\[
ds = r\, dt
\]

The outward unit normal vector:

\[
N(t) = \cos t\, \mathbf{i} + \sin t\, \mathbf{j}
\]

Evaluating $F \cdot N$

At each point:

$$
F = 5x\, \mathbf{i} + 5y\, \mathbf{j} = 5r \cos t\, \mathbf{i} + 5r \sin t\, \mathbf{j}
$$

Dot product:

$$
F \cdot N = (5r \cos t)(\cos t) + (5r \sin t)(\sin t) = 5r (\cos^2 t + \sin^2 t) = 5r
$$

Calculating the Line Integral for Flux

The flux:

$$
\Phi = \int_0^{2\pi} F \cdot N\, r\, dt = \int_0^{2\pi} 5r \times r\, dt = 5r^2 \int_0^{2\pi} dt = 5r^2 \times 2\pi = 10 \pi r^2
$$

This matches the double integral calculation via Green’s theorem, confirming the consistency.

Conclusion and Summary

The analysis of the vector field $F = 5x\, \mathbf{i} + 5y\, \mathbf{j}$ in conjunction with the outward unit normal vector N to a positively oriented circle illustrates key concepts in vector calculus. The normal vector N points outward, scaled by the radius, and plays an essential role in calculating flux across boundaries. The divergence of F being constant simplifies flux calculations, and the use of parameterization provides a straightforward method for evaluating line integrals.

Understanding these principles is crucial for applications in physics and engineering, such as fluid flow, electromagnetism, and other fields involving vector fields and boundary integrals. Mastery of the relationships between vector fields, normal vectors, and boundary orientations enables precise computation of fluxes, circulations, and other integral quantities vital for analyzing physical systems.

Additional Resources and Practice Problems

- Review articles on Green's theorem and its applications
- Practice calculating flux for different vector fields and boundary shapes
- Explore divergence theorem in three dimensions
- Simulate vector fields and their fluxes using computational tools like MATLAB or Wolfram Mathematica

By mastering these concepts, students and professionals can confidently approach problems involving vector calculus, boundary integrals, and field analysis.

Frequently Asked Questions

What is the vector field F given in the problem?

The vector field F is given by F = 5xi + 5yj.

What does N represent in the context of the circle?

N represents the outward unit normal vector to the positively oriented circle.

How do you find the outward unit normal vector N to a circle?

For a circle centered at the origin, the outward unit normal vector at a point is the normalized position vector from the center to that point on the circle.

What is the purpose of calculating the line integral of F over the circle?

The line integral helps evaluate the flux or circulation of the vector field F around the circle.

How is the outward unit normal vector N related to the circle's parametrization?

N is obtained by taking the position vector on the circle, then normalizing it to unit length, pointing outward from the circle's center.

Can the divergence theorem be applied in this context?

Yes, if considering the flux of F across the circle as part of a surface integral, the divergence theorem can relate the flux to the divergence of F over the enclosed area.

What is the divergence of the vector field F = 5xi + 5yj?

The divergence of F is ∂/∂x (5x) + ∂/∂y (5y) = 5 + 5 = 10.

How would you compute the flux of F across the circle using N?

The flux is computed as the line integral of F · N ds around the circle, where N is the outward unit normal vector and ds is the differential arc length.

Additional Resources

Let F = 5xi + 5yj and Let N Be The Outward Unit Normal Vector To The Positively Oriented Circle

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Introduction

In the realm of vector calculus and differential geometry, understanding the behavior of vector fields and their interactions with curves and surfaces is fundamental. One such exploration involves analyzing a vector field \(\mathbf{F} \) and its relation to a particular geometric object—in this case, a circle. Specifically, the vector field \(\mathbf{F} = 5x \mathbf{i} + 5y \mathbf{j} \) presents interesting properties when examined in conjunction with the outward unit normal vector \(\mathbf{N} \) to a positively oriented circle. This analysis not only deepens our conceptual grasp of vector fields and normal vectors but also has practical implications in physics, engineering, and applied mathematics, such as fluid flow, electromagnetism, and surface integrals.

This article offers a comprehensive review of the mathematical foundations, the geometric interpretations, and the integral calculus involved in studying $\mathbf{F}$ and $\mathbf{N}$ with respect to a circle. We will dissect the components systematically, beginning with the definition of the vector field, moving to the properties of the circle and its normal vectors, and culminating in the application of Green’s theorem, flux integrals, and related concepts.

Understanding the Vector Field $\mathbf{F} = 5x \mathbf{i} + 5y \mathbf{j}$

Nature and Properties of $\mathbf{F}$

The vector field $\mathbf{F}$ is defined as

$$
\mathbf{F} = 5x \mathbf{i} + 5y \mathbf{j}
$$

which can be interpreted geometrically and physically.

- Linear Field: The field varies linearly with position, meaning the magnitude and direction of $\mathbf{F}$ change proportionally with x and y.

- Radial Symmetry: Due to the form $5x$ and $5y$, $\mathbf{F}$ exhibits symmetry about the origin. Specifically, at any point (x, y), the vector points directly away from or toward the origin depending on the sign of x and y.

- Divergence and Curl:

- Divergence: $\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(5x) + \frac{\partial}{\partial y}(5y) = 5 + 5 = 10$

- Curl: $\nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}(5y) - \frac{\partial}{\partial y}(5x) \right) \mathbf{k} = (0 - 0) \mathbf{k} = 0$

This indicates that $\mathbf{F}$ is divergence-rich (sources everywhere) but irrotational, which has implications for flux calculations and potential flow modeling.

Physical Intuition

Imagine $\mathbf{F}$ as a field representing some outward flow emanating from the origin, with strength proportional to the distance from the center. Such fields are common in physics to model phenomena like electrostatic fields around a charge distribution or fluid flows from a source.

The Geometry of the Circle and Its Outward Normal Vector

Defining the Circle

The circle in question is typically described as

$$
x^2 + y^2 = r^2
$$

where r is the radius. For simplicity and generality, we assume the circle is centered at the origin, which aligns with the symmetry of $\mathbf{F}$.

Orientation and Positivity

In vector calculus, the positive orientation of a circle refers to counterclockwise traversal when viewed from above (along the positive z-axis). This orientation determines the direction of the outward normal vector and influences the sign conventions used in flux integrals.

Outward Unit Normal Vector $\mathbf{N}$

The outward normal vector to a circle at a point (x, y) on its circumference is:

$$
\mathbf{N} = \frac{\mathbf{r}}{\|\mathbf{r}\|} = \frac{x \mathbf{i} + y \mathbf{j}}{r}
$$

where

$$
\mathbf{r} = x \mathbf{i} + y \mathbf{j}
$$

and

$$
\|\mathbf{r}\| = \sqrt{x^2 + y^2} = r
$$

Thus, the outward unit normal vector at any point on the circle is:

$$
\boxed{
\mathbf{N} = \frac{x}{r} \mathbf{i} + \frac{y}{r} \mathbf{j}
}
$$

$$

This vector points directly away from the center of the circle, perpendicular to the tangent at that point, and outwardly oriented.

Significance in Surface and Line Integrals

The outward normal vector is crucial in evaluating flux integrals over the circle, which, in turn, helps determine how much of the vector field "flows" outward through the boundary.

Calculating Flux and Line Integrals

The Flux Integral

The flux of $\mathbf{F}$ across the circle boundary C is given by:

$$
\Phi = \oint_{C} \mathbf{F} \cdot \mathbf{N} \, ds
$$

where ds is the differential arc length along C.

Parameterizing the Circle

To evaluate the flux integral, a common approach is to parameterize the circle:

$$
x = r \cos t, \quad y = r \sin t, \quad t \in [0, 2\pi]
$$

Correspondingly,

$$
dx = -r \sin t \, dt, \quad dy = r \cos t \, dt
$$

The differential arc length:

$$
ds = r \, dt
$$

and the outward normal vector:

$$
\mathbf{N}(t) = \cos t \, \mathbf{i} + \sin t \, \mathbf{j}
$$

which aligns with the parametric points.

Computing $\mathbf{F} \cdot \mathbf{N}$

At each point:

$$
\mathbf{F}(x, y) = 5x \mathbf{i} + 5y \mathbf{j} = 5r \cos t \, \mathbf{i} + 5r \sin t \, \mathbf{j}
$$

and

$$
\mathbf{N}(t) = \cos t \, \mathbf{i} + \sin t \, \mathbf{j}
$$

Therefore,

$$
\mathbf{F} \cdot \mathbf{N} = (5r \cos t)(\cos t) + (5r \sin t)(\sin t) = 5r (\cos^2 t + \sin^2 t) = 5r
$$

Computing the Flux

The flux integral becomes:

$$
\Phi = \int_{0}^{2\pi} \mathbf{F} \cdot \mathbf{N} \, ds = \int_{0}^{2\pi} 5r \times r \, dt = 5 r^2 \int_{0}^{2\pi} dt = 5 r^2 \times 2\pi = 10 \pi r^2
$$

This result indicates that the flux of $\mathbf{F}$ across the circle scales with the area of the circle, consistent with the divergence theorem.

Divergence Theorem and Its Application

Statement of the Divergence Theorem

The divergence theorem states:

$$
\iint_{S} (\nabla \cdot \mathbf{F}) \, dA = \oint_{C} \mathbf{F} \cdot \mathbf{N} \, ds
$$

where:

- \(S \) is the surface (area) enclosed by the boundary \(C \),
- \(\mathbf{N} \) is the outward normal to \(C \).

Given the divergence \(\nabla \cdot \mathbf{F} = 10 \), the total flux through the circle should match:

\[
\iint_{S} 10 \, dA = 10 \times \pi r^2
\]

which aligns with the previous flux calculation:

\[
\boxed{
\oint_{C} \mathbf{F} \cdot \mathbf{N} \, ds = 10 \pi r^2
}
\]

This consistency validates the earlier analysis and exemplifies divergence theorem application.

Additional Analytical Perspectives

Line Integral of \(\mathbf{F} \) Around \(C \)

While the flux integral measures outward flow, the line integral of \(\mathbf{F} \) around \(C \):

\[
\oint_{C} \mathbf{F} \cdot d\mathbf{r}
\]

has different implications, often relating to circulation.

Using the parameterization:

\[
d\mathbf{r} = dx \, \mathbf{i} + dy \, \mathbf{j}
\]

and the earlier expressions,

1 Point Let F 5xi 5yj And Let N Be The Outward Unit Normal Vector To The Positively Oriented Circle

1 Point Let F 5xi 5yj And Let N Be The Outward Unit Normal Vector To The Positively Oriented

Circle

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