2- Find The Solution Of Laplace's Equation In Spherical Coordinates, Where U(r, 8), Where R Is The Radius

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Laplace's equation is a fundamental second-order partial differential equation that appears extensively in fields such as electrostatics, gravitation, fluid dynamics, and thermal conduction. Solving Laplace's equation in various coordinate systems helps in addressing boundary value problems with different geometries. In this article, we focus on solving Laplace's equation in spherical coordinates, where the potential function $U(r, \theta, \phi)$ depends on the radial distance r, the polar angle θ, and the azimuthal angle ϕ. Specifically, we analyze the case where the potential function is independent of ϕ, i.e., $U(r, \theta)$, highlighting solutions involving the radius R.

Understanding Laplace's Equation in Spherical Coordinates

Laplace's equation in three-dimensional Cartesian coordinates is expressed as:

$$
\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0
$$

However, many physical problems involve spherical symmetry, making spherical coordinates more suitable. The transformation from Cartesian to spherical coordinates is given by:

$$
\begin{cases}
x = r \sin \theta \cos \phi \\
y = r \sin \theta \sin \phi \\
z = r \cos \theta
\end{cases}
$$

where:
- $r \geq 0$ is the radial distance from the origin,

- $0 \leq \theta \leq \pi$ is the polar (colatitude) angle measured from the positive z-axis,
- $0 \leq \phi \leq 2\pi$ is the azimuthal angle in the xy-plane.

In spherical coordinates, Laplace's equation takes the form:

$$
\nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2} = 0
$$

Assumptions and Simplifications

To simplify the problem, consider the case where the potential $U(r, \theta)$ does not depend on the azimuthal angle ϕ. This assumption is valid in problems with azimuthal symmetry, such as a sphere with uniform boundary conditions around the z-axis. Under this assumption, the term involving $\partial^2 U / \partial \phi^2$ drops out, and Laplace's equation reduces to:

$$
\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) = 0
$$

This form allows separation of variables, leading to solutions expressed as products of radial and angular functions.

Method of Separation of Variables

The key approach to solving Laplace's equation in spherical coordinates is the method of separation of variables, assuming solutions of the form:

$$
U(r, \theta) = R(r) \Theta(\theta)
$$

Substituting into the simplified Laplace's equation, dividing through by \(

$R(r) \Theta(\theta)$, and rearranging yields:

$$
\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{d R}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d \theta} \left(\sin \theta \frac{d \Theta}{d \theta} \right) = 0
$$

Since the variables are separated, each term must equal a constant, say $l(l+1)$, leading to two ordinary differential equations:

1. Angular Equation (Associated Legendre Equation):

$$
\frac{1}{\sin \theta} \frac{d}{d \theta} \left(\sin \theta \frac{d \Theta}{d \theta} \right) + l(l+1) \Theta = 0
$$

2. Radial Equation:

$$
\frac{d}{dr} \left(r^2 \frac{d R}{dr} \right) - l(l+1) R = 0
$$

Solution to the Angular Equation

The angular differential equation is known as the Legendre equation, with solutions called Legendre polynomials $P_l(\cos \theta)$:

$$
\Theta(\theta) = P_l(\cos \theta)
$$

where $l = 0, 1, 2, \ldots$. These polynomials are orthogonal over the interval $\theta \in [0, \pi]$ and form a complete set for representing functions with azimuthal symmetry.

Key properties of Legendre polynomials:

- $P_0(\cos \theta) = 1$
- $P_1(\cos \theta) = \cos \theta$
- $P_2(\cos \theta) = \frac{1}{2} (3 \cos^2 \theta - 1)$

Solution to the Radial Equation

The radial differential equation simplifies to:

$$
r^2 \frac{d^2 R}{dr^2} + 2 r \frac{d R}{dr} - l(l+1) R = 0
$$

which is an Euler-Cauchy equation. Its general solution is:

$$
R(r) = A_l r^l + \frac{B_l}{r^{l+1}}
$$

where A_l and B_l are constants determined by boundary conditions.

General Solution in Spherical Coordinates

Combining the solutions for the angular and radial parts, the general solution for $U(r, \theta)$ (assuming azimuthal symmetry) is:

$$
U(r, \theta) = \sum_{l=0}^\infty \left[A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l(\cos \theta)
$$

This series expansion allows modeling of various boundary conditions.

Applying Boundary Conditions

In physical problems, boundary conditions specify the behavior of U at certain surfaces or at infinity. Some typical boundary conditions include:

- Potential specified at a spherical surface $r = R$:

$$
U(R, \theta) = U_0(\theta)
$$

- Potential finite at the origin $r = 0$:

$$
U(r, \theta) \text{ remains finite as } r \to 0
$$

- Potential vanishing at infinity:

$$
U(r, \theta) \to 0 \quad \text{as} \quad r \to \infty
$$

Depending on these, certain coefficients A_l and B_l are set to zero to satisfy physical constraints.

Special Case: Potential Inside a Sphere

Consider the classical problem of finding potential inside a spherical region $r \leq R$ with boundary $U(R, \theta) = U_0(\theta)$. Since the potential must be finite at $r = 0$, the terms involving $r^{-(l+1)}$ are discarded ($B_l = 0$). The solution simplifies to:

$$
U(r, \theta) = \sum_{l=0}^\infty A_l r^l P_l(\cos \theta)
$$

The coefficients A_l are determined by projecting the boundary condition onto the Legendre polynomials:

$$
A_l = \frac{1}{R^l} \frac{2l + 1}{2} \int_0^\pi U_0(\theta) P_l(\cos \theta) \sin \theta \, d\theta
$$

Special Case: Potential Outside a Sphere

If the potential is specified outside the sphere $r \geq R$, and it must vanish at infinity, the terms involving r^l are discarded ($A_l = 0$). The solution becomes:

$$
U(r, \theta) = \sum_{l=0}^\infty \frac{B_l}{r^{l+1}} P_l(\cos \theta)
$$

Similarly,

Frequently Asked Questions

What is Laplace's equation in spherical coordinates for a function $U(r, \theta, \varphi)$?

Laplace's equation in spherical coordinates is given by

$$\nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial U}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 U}{\partial \varphi^2} = 0.$$

For problems where U depends only on r and θ (i.e., axisymmetric), the equation simplifies accordingly.

How do you separate variables when solving Laplace's equation in spherical coordinates?

You assume a solution of the form $U(r, \theta, \varphi) = R(r) \times \Theta(\theta) \times \Phi(\varphi)$. For azimuthally symmetric problems, U depends only on r and θ, reducing the solution to $U(r, \theta) = R(r) \times \Theta(\theta)$. Substituting into Laplace's equation leads to separate ordinary differential equations for $R(r)$ and $\Theta(\theta)$.

What are the general solutions for the radial part R(r) in Laplace's equation?

The radial solutions are typically of the form $R(r) = A r^l + B r^{-(l+1)}$, where l is a non-negative integer. The choice of solution depends on boundary conditions, such as behavior at the origin or at infinity.

How is the angular part Θ(θ) of the solution expressed in spherical harmonics?

The angular part $\Theta(\theta)$ corresponds to associated Legendre functions $P_l^m(\cos\theta)$. For axisymmetric problems (no φ dependence), $\Theta(\theta)$ reduces to Legendre polynomials $P_l(\cos\theta)$.

What boundary conditions are typically used to solve Laplace's equation in spherical coordinates with radius R?

Common boundary conditions include specifying U on the spherical surface $r = R$, such as $U(R, \theta) = f(\theta)$, or conditions at $r = 0$ or $r \to \infty$. These help determine the coefficients in the general solution.

Can you provide an example solution of Laplace's equation in spherical coordinates for a given boundary condition?

Yes. For example, if $U(R, \theta) = P_l(\cos \theta)$, then the solution inside the sphere ($r < R$) is $U(r, \theta) = (r^l / R^l) P_l(\cos \theta)$. This satisfies Laplace's equation and matches the boundary condition.

How does the solution change if the boundary condition is specified on the sphere's surface?

The coefficients in the series expansion are determined by projecting the boundary condition onto the basis functions (Legendre polynomials). The solution inside the sphere involves only the terms consistent with the boundary data.

What is the significance of the Legendre polynomials in solving Laplace's equation in spherical coordinates?

Legendre polynomials form the angular part of the solution in axisymmetric problems. They ensure the solution satisfies the angular part of Laplace's equation and match the boundary conditions on the sphere.

Are there special cases or symmetries that simplify solving Laplace's equation in spherical coordinates?

Yes. For axisymmetric problems (no φ dependence), the solution reduces to Legendre polynomials. Spherical symmetry (U depending only on r) simplifies the problem further, often leading to solutions involving only r^l and $r^{-(l+1)}$.

What methods are commonly used to find solutions of Laplace's equation in spherical coordinates with given boundary conditions?

Methods include separation of variables, expansion in spherical harmonics or Legendre polynomials, and applying boundary conditions to determine expansion coefficients. These techniques leverage the orthogonality of the basis functions to simplify the problem.

Additional Resources

2. Find The Solution Of Laplace's Equation In Spherical Coordinates, Where $U(r, \theta)$, Where R Is The Radius

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Introduction

Laplace's equation stands as a cornerstone in the field of potential theory, mathematical physics, and electromagnetism. Its solutions describe steady-state solutions to Laplace's differential operator, which appears ubiquitously in problems involving electrostatics, gravitational fields, and incompressible fluid flow. When dealing with problems exhibiting spherical symmetry, it becomes essential to adapt the Cartesian form of Laplace’s equation into spherical coordinates (r, θ, ϕ), where r is the radial distance, θ the polar angle, and ϕ the azimuthal angle.

This article aims to carefully derive the general solution to Laplace's equation in spherical coordinates, particularly focusing on functions $U(r, \theta)$ where the potential depends solely on the radius r and the polar angle θ, with the boundary surface at radius R. Such problems are fundamental in modeling phenomena like the potential outside a spherical charge distribution, the temperature distribution within spherical objects, or gravitational potential outside a spherical mass.

The Significance of Laplace's Equation in Spherical Coordinates

Laplace's equation in three dimensions is expressed as:

$$
\nabla^2 U = 0
$$

In Cartesian coordinates, it reads:

$$
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0
$$

However, for problems with spherical symmetry or partial symmetry, spherical coordinates are more natural. The Laplacian in spherical coordinates is:

$$
\nabla^2 U = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial U}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 U}{\partial \phi^2}
$$

Given the problem's symmetry, solutions often depend only on r and

θ, ignoring ϕ dependence, especially in axisymmetric problems.

Formulating the Boundary Value Problem

Consider a problem where the potential $U(r, \theta)$ satisfies Laplace's equation within a spherical domain $0 < r < R$, with boundary conditions specified on the sphere $r=R$. The typical boundary conditions could be:

- Dirichlet boundary condition: $U(R, \theta) = f(\theta)$,
- Neumann boundary condition: $\left.\frac{\partial U}{\partial r}\right|_{r=R} = g(\theta)$.

The goal is to find $U(r, \theta)$ satisfying Laplace's equation and these boundary conditions.

Separation of Variables: A Methodical Approach

The classical approach to solving Laplace's equation in spherical coordinates involves separation of variables, assuming a solution of the form:

$$
U(r, \theta) = R(r) \Theta(\theta)
$$

This leads to splitting the partial differential equation into two ordinary differential equations, one in r and another in θ.

Deriving the General Solution

Step 1: Substituting the Separable Solution

Substituting $U(r, \theta) = R(r) \Theta(\theta)$ into Laplace's equation yields:

$$
\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d R}{dr} \right) \Theta(\theta) + \frac{1}{r^2 \sin \theta} \frac{d}{d \theta} \left(\sin \theta \frac{d \Theta}{d \theta} \right) R(r) = 0
$$

Dividing through by $R(r) \Theta(\theta)$:

$$
\frac{1}{R} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d R}{dr} \right) +

$$
\frac{1}{\Theta} \frac{1}{\sin \theta} \frac{d}{d \theta} \left(\sin \theta \frac{d \Theta}{d \theta} \right) = 0
$$

Since the left side depends only on r and the right only on θ, both must equal a constant, conventionally denoted as $\ell(\ell + 1)$.

Step 2: Angular Equation - Legendre Differential Equation

The angular part becomes:

$$
\frac{1}{\sin \theta} \frac{d}{d \theta} \left(\sin \theta \frac{d \Theta}{d \theta} \right) + \ell(\ell + 1) \Theta = 0
$$

This is the Legendre differential equation, whose solutions are the Legendre polynomials $P_\ell(\cos \theta)$.

Key properties:

- $\ell = 0, 1, 2, \ldots$,
- $\Theta(\theta) = P_\ell(\cos \theta)$.

Step 3: Radial Equation

The radial equation becomes:

$$
\frac{d}{dr} \left(r^2 \frac{d R}{dr} \right) - \ell (\ell + 1) R = 0
$$

Or explicitly:

$$
r^2 \frac{d^2 R}{dr^2} + 2r \frac{d R}{dr} - \ell (\ell + 1) R = 0
$$

This is an Euler-Cauchy equation with general solutions:

$$
R(r) = A_\ell r^\ell + B_\ell r^{-(\ell + 1)}
$$

where A_ℓ and B_ℓ are constants determined by boundary conditions.

The General Solution in Spherical Coordinates

Combining the radial and angular parts, the general solution to Laplace's equation in spherical coordinates (assuming axisymmetry, no ϕ dependence) is:

$$
U(r, \theta) = \sum_{\ell=0}^{\infty} \left[A_\ell r^\ell + B_\ell r^{-(\ell + 1)} \right] P_\ell(\cos \theta)
$$

This series expansion converges within the domain of interest and is tailored based on boundary conditions.

Applying Boundary Conditions for a Sphere of Radius R

Suppose the potential is specified on the surface $r=R$:

$$
U(R, \theta) = f(\theta)
$$

Expressed as a Legendre series:

$$
f(\theta) = \sum_{\ell=0}^{\infty} f_\ell P_\ell(\cos \theta)
$$

Matching terms at $r=R$:

$$
f_\ell = A_\ell R^\ell + B_\ell R^{-(\ell + 1)}
$$

If the potential must be finite at the origin ($r=0$), then:

$$
B_\ell = 0
$$

since $r^{-(\ell+1)}$ diverges as $r \to 0$. Conversely, if the potential is specified outside the sphere, the solution must decay at infinity, leading to:

$$
A_\ell = 0
$$

depending on the problem's physical context.

Special Cases and Physical Interpretations

- Potential outside the sphere (for $r \ge R$):

$$
U(r, \theta) = \sum_{\ell=0}^\infty C_\ell r^{-(\ell + 1)} P_\ell(\cos \theta)
$$

- Potential inside the sphere (for $0 \le r \le R$):

$$
U(r, \theta) = \sum_{\ell=0}^\infty D_\ell r^\ell P_\ell(\cos \theta)
$$

The coefficients C_ℓ or D_ℓ are determined via boundary conditions and physical constraints.

Solving for Specific Boundary Conditions

To demonstrate the applicability, consider a boundary condition:

$$
U(R, \theta) = V_0 P_1(\cos \theta) = V_0 \cos \theta
$$

which implies only $\ell=1$ term survives. The solution simplifies to:

$$
U(r, \theta) = \left(A_1 r + B_1 r^{-2} \right) P_1(\cos \theta)
$$

Matching at $r=R$:

$$
V_0 \cos \theta = (A_1 R + B_1 R^{-2}) \cos \theta
$$

Leading to:

$$
A_1 R + B_1 R^{-2} = V_0
$$

Choosing the appropriate physical

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