The Function F Is Defined By F (x) = Sin(x2). What Are The First Four Nonzero Terms Of The Taylor Series

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Understanding the Taylor series expansion of functions is fundamental in calculus, especially when approximating complex functions with polynomial expressions. In this article, we will explore the Taylor series expansion of the function $F(x) = \sin(x^2)$ and identify its first four nonzero terms. This analysis provides insight into how the function behaves near a specific point, typically around $x=0$, and demonstrates the process of deriving Taylor series expansions for composite functions.

Introduction to Taylor Series and Its Relevance

What Is a Taylor Series?

A Taylor series is an infinite sum of terms constructed from the derivatives of a function at a single point, usually centered at a. It expresses a function as an approximation by polynomials, which becomes increasingly accurate as more terms are included.

Mathematically, the Taylor series of a function $f(x)$ centered at a is given by:

$$
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
$$

where $f^{(n)}(a)$ denotes the nth derivative evaluated at a.

Why Use Taylor Series?

Taylor series are particularly useful because:
- They simplify complicated functions into polynomials for analysis and computation.
- They facilitate numerical methods for function approximation.
- They help analyze the local behavior of functions near a specific point.

Analyzing the Function $F(x) = \sin(x^2)$

Choosing the Expansion Point

Typically, Taylor series are expanded around $a=0$ (Maclaurin series) for simplicity unless specified otherwise. For $F(x) = \sin(x^2)$, we will compute its Maclaurin series, i.e., expansion around $x=0$.

Understanding the Function

The function $F(x) = \sin(x^2)$ is a composition of the sine function and the quadratic function x^2. Recognizing this helps us apply known series expansions and differentiation rules more efficiently.

Deriving the Taylor Series of $F(x) = \sin(x^2)$

Step 1: Recall the Series for $\sin(z)$

The Maclaurin series expansion for $\sin(z)$ is:

$$
\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots
$$

which converges for all real z.

Step 2: Substitute $z = x^2$

Since $F(x) = \sin(x^2)$, substitute $z = x^2$ into the sine series:

$$
F(x) = \sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots
$$

Simplify each term:

$$
F(x) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots
$$

Step 3: Write the Series Terms Explicitly

Expressing the first few terms explicitly:

$$
F(x) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} + \cdots
$$

Here, note that:
- $3! = 6$
- $5! = 120$
- $7! = 5040$

Identifying the First Four Nonzero Terms

The series expansion clearly shows the pattern of nonzero terms:

1. x^2
2. $- \frac{x^6}{6}$
3. $+ \frac{x^{10}}{120}$
4. $- \frac{x^{14}}{5040}$

These are the first four nonzero terms of the Taylor (Maclaurin) series expansion of $F(x) = \sin(x^2)$.

Summary of the First Four Nonzero Terms

$$
\boxed{
F(x) \approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}
}
$$

This polynomial approximation is valid near $x=0$ and provides a good estimate of $F(x)$ for small values of x.

Understanding the Significance of These Terms

Behavior Near $x=0$

The leading term x^2 indicates that near zero, $F(x)$ behaves approximately like x^2. Since higher-order terms involve higher powers of x, their influence diminishes as x approaches zero.

Application in Approximation and Computation

These terms are crucial in:
- Numerical methods, such as polynomial approximations for computing $\sin(x^2)$.
- Analyzing the function's local behavior.
- Solving differential equations involving $\sin(x^2)$.

Further Exploration: Derivatives and Remainder Terms

Higher-Order Derivatives at $x=0$

While the series expansion provides a straightforward way to generate the terms, understanding the derivatives explicitly can deepen comprehension. For the Maclaurin series, derivatives at zero determine the coefficients.

Remainder Term and Series Convergence

The series converges for all real x because $\sin(z)$ converges for all z. The remainder term's size determines the approximation's accuracy for finite n.

Conclusion

In this article, we examined the function $F(x) = \sin(x^2)$ and derived its Taylor series expansion around $x=0$. The first four nonzero terms are:

$$
x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}
$$

These terms provide a polynomial approximation useful in various applications, including numerical analysis and understanding the function's local behavior. Recognizing the pattern in the series helps in extending the approximation further and analyzing related functions.

Additional Resources

- Textbooks on Calculus: For a more comprehensive understanding of Taylor and Maclaurin series.
- Mathematical Software: Tools like WolframAlpha, MATLAB, or Mathematica can compute series expansions automatically.
- Online Calculus Courses: Many platforms offer courses covering Taylor series and their applications.

Understanding how to expand functions like $\sin(x^2)$ into Taylor series not only enhances your grasp of calculus but also equips you with practical tools for scientific computation and mathematical modeling.

Frequently Asked Questions

What is the general approach to finding the first four nonzero

terms of the Taylor series for $F(x) = \sin(x^2)$?

To find the first four nonzero terms, expand sin(t) into its Taylor series, then substitute $t = x^2$, and simplify the resulting terms.

What is the Maclaurin series expansion for sin(t)?

The Maclaurin series for sin(t) is $t - t^3/3! + t^5/5! - t^7/7! + \ldots$, valid for all real t.

How do you substitute $t = x^2$ into the series for sin(t)?

Replace every occurrence of t in the series with x^2, resulting in terms like x^2, $x^6/3!$, $x^{10}/5!$, etc.

What are the first four nonzero terms of the Taylor series for $F(x) = \sin(x^2)$?

The first four nonzero terms are $x^2 - x^6/3! + x^{10}/5! - x^{14}/7!$.

How do factorial terms apply when writing the explicit coefficients of the series?

Use factorial values in the denominators: $3! = 6$, $5! = 120$, $7! = 5040$, to write the coefficients explicitly.

Can you write out the first four nonzero terms explicitly with factorials included?

Yes. The first four terms are: $x^2 - x^6/6 + x^{10}/120 - x^{14}/5040$.

Additional Resources

The Function F Is Defined By F (x) = Sin(x2). What Are The First Four Nonzero Terms Of The Taylor Series

In the realm of calculus and mathematical analysis, understanding how functions behave near specific points is fundamental. One of the most powerful tools mathematicians use to approximate functions locally is the Taylor series. For the function F(x) = sin(x^2), deriving its Taylor series expansion provides insight into its behavior around a particular point—most commonly at x=0. This article explores the process of finding the first four nonzero terms of the Taylor series for F(x) = sin(x^2), breaking down the steps involved, and elucidating the significance of these terms in approximation theory.

Understanding the Taylor Series Concept

Before delving into the specifics of the function $F(x) = \sin(x^2)$, it is essential to grasp the fundamental concept of Taylor series.

What Is a Taylor Series?

A Taylor series is an infinite sum of terms that represents a function as an approximation around a specific point, often $x=0$ (called the Maclaurin series). It is expressed as:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

where:

- $f^{(n)}(a)$ is the n-th derivative of f evaluated at point a,
- $n!$ is the factorial of n,
- a is the point around which the series is expanded.

When $a=0$, the series is called a Maclaurin series.

Why Use a Taylor Series?

Taylor series enable us to approximate complex functions with polynomial expressions, which are easier to compute and analyze. For small values of x close to the expansion point, these polynomials provide highly accurate approximations. This is especially useful in engineering, physics, and computational mathematics where exact solutions can be difficult or impossible to derive.

Formulating the Taylor Series for $F(x) = \sin(x^2)$

To find the Taylor series for $F(x) = \sin(x^2)$, the process involves calculating derivatives of the function at $x=0$ and then plugging these into the Taylor series formula.

Step 1: Recognize the Basic Series

The sine function has a well-known Taylor series expansion at $x=0$:

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

where t is any real number.

In our case, since $F(x) = \sin(x^2)$, we can substitute $t = x^2$:

$$ \sin(x^2) = x^2 - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots $$

This substitution simplifies the derivation, as we leverage the known series expansion of $\sin(t)$.

Step 2: Expand the Series

Applying the substitution:

$$ \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots $$

Now, explicitly writing out the factorial values:

$$ 3! = 6 $$
$$ 5! = 120 $$
$$ 7! = 5040 $$

the series becomes:

$$ \sin(x^2) = x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040} + \cdots $$

Identifying the First Four Nonzero Terms

The series derived above naturally presents the nonzero terms in order of increasing powers of x. Let's examine these terms carefully.

First Four Nonzero Terms

1. First term: $ x^2 $

2. Second term: $ - \frac{x^6}{6} $

3. Third term: $ + \frac{x^{10}}{120} $

4. Fourth term: $ - \frac{x^{14}}{5040} $

These four terms are the earliest non-zero components of the Taylor series expansion of $F(x)$ around $x=0$, providing the primary approximation for small x.

Why These Terms Are Significant

- Order of terms: They represent the polynomial parts that approximate $\sin(x^2)$ near zero.
- Alternating signs: The series oscillates, with positive and negative terms, reflecting the oscillatory nature of the sine function.
- Impact of exponents: The powers grow rapidly, indicating that higher-order terms contribute less for small x, making the initial terms particularly useful for approximation near zero.

Interpreting the Approximation and Its Applications

Understanding the first four nonzero terms of the Taylor series for $\sin(x^2)$ offers practical benefits in various scientific and engineering contexts.

Approximate Behavior Near Zero

For values of x close to zero, higher powers of x become negligible, and the truncated series provides a close approximation:

$$\sin(x^2) \approx x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}$$

This approximation allows mathematicians and engineers to analyze the behavior of the function without resorting to more complex calculations.

Applications in Physics and Engineering

- Wave phenomena: Approximating oscillations where $\sin(x^2)$ models certain wave functions or signal modulations.
- Quantum mechanics: When dealing with potential functions or wave functions expanded around equilibrium points.
- Numerical methods: Developing algorithms that rely on polynomial approximations for functions that are otherwise computationally intensive.

Limitations of the Series

While the Taylor series provides a powerful approximation near the expansion point, its accuracy diminishes as x moves further from zero. The higher-order terms become significant, and the polynomial no longer accurately reflects the function's behavior. Therefore, in practical applications, the range of x should be considered, and the series truncated accordingly.

Conclusion: The Power of Taylor Series in Function Approximation

The derivation of the first four nonzero terms of the Taylor series for $F(x) = \sin(x^2)$ exemplifies the elegant way in which complex functions can be approximated by simple polynomials. Recognizing the pattern—substituting $t = x^2$ into the well-known sine expansion—streamlines the process and reveals the structure of the series succinctly. The resulting terms, x^2, $-x^6/6$, $+x^{10}/120$, and $-x^{14}/5040$, serve as foundational building blocks for approximating the function's behavior near zero.

This approach not only enhances our understanding of the function's local behavior but also underscores the broader utility of Taylor series in scientific computation, modeling, and theoretical analysis. As functions grow more complex, the ability to approximate them with polynomial expressions remains a cornerstone of applied mathematics, offering clarity and computational efficiency in diverse fields.

In summary, the first four nonzero terms of the Taylor series for $F(x) = \sin(x^2)$ are:

$$x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \frac{x^{14}}{5040}$$

These terms encapsulate the essence of the function's local behavior, demonstrating the power and elegance of Taylor series in mathematical analysis and beyond.

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