The Line 10x + Py = Q Is A Tangent At The Point (5, 4) In Another Circle With Centre(0,0).Find The Value

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Understanding the properties of tangents and circles is fundamental in coordinate geometry. In this article, we will explore how to determine the value of P and Q in the given line equation, which is tangent to a circle centered at the origin (0, 0), touching it at the point (5, 4). This problem involves concepts such as the equation of a tangent line, the circle's equation, and the geometric relationship between tangents and circles.

Introduction to the Problem

The problem statement is as follows:

> The line 10x + Py = Q is tangent to a circle with center at (0, 0) and passes through the point (5, 4). Find the values of P and Q.

Key points to note:

- The line is given in the form 10x + Py = Q.
- The line touches the circle at exactly one point, (5, 4).
- The circle has its center at the origin (0, 0).

Our goal is to find the unknown parameters P and Q based on this information.

Understanding the Geometric Context

Before solving for P and Q, it's important to understand some fundamental concepts.

The Equation of a Circle Centered at the Origin

The general equation of a circle with center at (0, 0) and radius R is:

$$ x^2 + y^2 = R^2 $$

Since the circle's radius is not given, it is an unknown, but the key is that the point (5, 4) lies on the circle, and the line is tangent to the circle at this point.

The Equation of the Tangent Line

The line is:

$$ 10x + Py = Q $$

This line passes through the point (5, 4). Since it is tangent to the circle, it touches the circle at exactly one point, which is (5, 4).

Step-by-Step Solution Approach

To find P and Q, we will follow these steps:

1. Use the point (5, 4) to relate P and Q.
2. Use the tangent condition: the line is tangent to the circle at (5, 4).
3. Find the circle's radius using the point (5, 4).
4. Derive the condition for the line being tangent to the circle at (5, 4).

Step 1: Substituting the Point (5, 4) into the Line Equation

Since the line passes through (5, 4):

$$
10 \times 5 + P \times 4 = Q
$$
$$
50 + 4P = Q
$$

This gives us a relationship between P and Q:

$$
Q = 50 + 4P \quad \text{(Equation 1)}
$$

Step 2: Find the Radius of the Circle

Because the circle passes through (5, 4), and its center is at (0, 0), the radius R is:

$$
R = \sqrt{(5 - 0)^2 + (4 - 0)^2} = \sqrt{25 + 16} = \sqrt{41}
$$

Thus, the circle's equation is:

$$
x^2 + y^2 = 41
$$

Step 3: Tangency Condition

A line is tangent to a circle if the perpendicular distance from the circle's center to the line is equal to the radius R.

The distance d from the center (0, 0) to the line $10x + Py = Q$ is:

$$
d = \frac{|Q|}{\sqrt{10^2 + P^2}} = \frac{|Q|}{\sqrt{100 + P^2}}
$$

Since the line is tangent at (5, 4), this distance must be equal to the radius $\sqrt{41}$:

$$
\frac{|Q|}{\sqrt{100 + P^2}} = \sqrt{41}
$$

Using Equation 1 for Q:

$$
\frac{|50 + 4P|}{\sqrt{100 + P^2}} = \sqrt{41}
$$

Step 4: Solve for P

Squaring both sides to eliminate the absolute value:

$$
\frac{(50 + 4P)^2}{100 + P^2} = 41
$$

Multiply both sides by $100 + P^2$:

$$
(50 + 4P)^2 = 41 (100 + P^2)
$$

Expand the left side:

$$
(50)^2 + 2 \times 50 \times 4P + (4P)^2 = 4100 + 41 P^2
$$

$$
2500 + 400 P + 16 P^2 = 4100 + 41 P^2
$$

Bring all to one side:

$$
2500 + 400 P + 16 P^2 - 4100 - 41 P^2 = 0
$$

Simplify:

$$
(2500 - 4100) + 400 P + (16 P^2 - 41 P^2) = 0
$$

$$
-1600 + 400 P - 25 P^2 = 0
$$

Divide entire equation by 25 to simplify:

$$
-64 + 16 P - P^2 = 0
$$

Rewrite:

$$
-P^2 + 16 P - 64 = 0
$$

Multiply through by -1 to standardize:

$$
P^2 - 16 P + 64 = 0
$$

Step 5: Solve the Quadratic Equation for P

The quadratic:

\[
P^2 - 16 P + 64 = 0
\]

Discriminant:

\[
\Delta = (-16)^2 - 4 \times 1 \times 64 = 256 - 256 = 0
\]

Since the discriminant is zero, there is one unique solution:

\[
P = \frac{16}{2} = 8
\]

Step 6: Find Q Using P

Recall from Equation 1:

\[
Q = 50 + 4 P
\]

Substitute \(P = 8\):

\[
Q = 50 + 4 \times 8 = 50 + 32 = 82
\]

Final Answer: The Values of P and Q

\[
\boxed{

\textbf{P} = 8 \quad \text{and} \quad \textbf{Q} = 82
}
\]

These values satisfy the conditions of the problem: the line \(10x + 8 y = 82\) is tangent to the circle centered at (0,0) and passes through (5, 4), touching the circle at exactly one point.

Summary and Key Takeaways

- The point (5, 4) lies on both the circle and the tangent line.
- The tangent condition involves the perpendicular distance from the circle's center to the line.
- Solving the quadratic equation yielded a unique P value due to the discriminant being zero, indicating a single tangent line.
- Calculating Q using the relationship with P confirmed the line passes through (5, 4).

Additional Tips for Similar Problems

- Always verify whether the line passes through the given point.
- Use the perpendicular distance formula to enforce tangency conditions.
- When quadratic equations yield a discriminant of zero, it indicates a unique solution, often associated with tangent lines.
- Remember that the circle's radius can be found from the distance between the center and any point on the circle.

Conclusion

By carefully applying coordinate geometry principles—substituting points, applying the tangent condition, and solving the resulting quadratic equations—we determined the specific values of P and Q for the given problem. The key takeaway is understanding how the geometric properties of tangents and circles translate into algebraic equations, enabling precise solutions in coordinate geometry challenges.

References:

- Coordinate Geometry textbooks
- Mathematical problem-solving guides
- Online resources on circle equations and tangent line properties

Frequently Asked Questions

How do you determine the value of Q for the line 10x + Py = Q to be tangent to a circle centered at (0,0)?

To find Q, substitute the point of tangency (5,4) into the line equation and ensure the line intersects the circle at exactly one point, which leads to solving for Q using the coordinates (Q = 105 + P4). Additionally, the perpendicular distance from the circle's center to the line must equal the radius of the circle for tangency, providing a method to find Q.

What is the significance of the point (5, 4) in determining the tangent line to the circle centered at (0, 0)?

The point (5, 4) is the point of tangency, meaning the line touches the circle exactly at this point. This allows us to use its coordinates to find the line's equation and the value of Q, ensuring the line is tangent at that specific point.

How can the condition for tangency help in finding the value of P in the line equation 10x + Py = Q?

Since the line is tangent to the circle centered at (0,0), the perpendicular distance from the center to the line must be equal to the circle's radius. Using the point (5, 4) and the tangent condition, we can relate P and Q to find their values, especially P, once Q is determined.

Is the radius of the circle needed to find the value of Q in the tangent line problem? Why or why not?

Yes, the radius is needed because the tangent condition requires the perpendicular distance from the circle's center to the line to equal the radius. Knowing the radius allows us to set up equations to solve for Q and P accordingly.

What steps are involved in solving for Q given the tangent point (5, 4) and the line 10x + Py = Q?

First, substitute (5, 4) into the line to express Q as Q = 105 + P4. Then, use the tangent condition that the perpendicular distance from the center (0,0) to the line equals the radius (which can be found or assumed). This leads to solving for P and Q simultaneously.

Can the value of P be directly obtained once Q is known? How?

Yes, once Q is determined using the point (5, 4), you can use the tangent condition or additional geometric relations to solve for P. For example, if the radius is known, you can set up the perpendicular distance formula and solve for P accordingly.

Additional Resources

The Line 10x + Py = Q Is A Tangent At The Point (5, 4) In Another Circle With Centre(0,0). Find The Value

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Introduction

Mathematics, especially the geometry of circles and tangents, continues to be a cornerstone of analytical problem solving. The problem at hand involves a line expressed in the form 10x + Py = Q, which is tangent to a circle centered at the origin (0, 0), touching it at a specific point (5, 4). The primary objective is to determine the value of P (and potentially Q) that satisfies these conditions. This problem not only tests understanding of tangent lines and circles but also explores the interplay between algebraic and geometric representations.

In this comprehensive review, we will dissect the problem step-by-step, exploring the underlying principles, derivations, and methodologies to arrive at the solution. The investigation emphasizes clarity, logical progression, and the application of fundamental geometric concepts.

Geometric and Algebraic Foundations

The Circle with Center at the Origin

Given a circle with center at (0, 0), its standard equation is:

\[
x^2 + y^2 = r^2
\]

where \(r \) is the radius. The problem, however, does not specify the radius explicitly; instead, it focuses on a line tangent to this circle at a known point.

The Tangent Line

The line's equation is provided in the form:

\[

$$
10x + Py = Q
$$

which can be rewritten as:

$$
10x + Py - Q = 0
$$

This is a linear equation in the standard form $Ax + By + C = 0$, with:

- $A = 10$
- $B = P$
- $C = -Q$

Step 1: Understanding the Tangency Condition

Geometric interpretation

For the line to be tangent to the circle, it must touch the circle at exactly one point. The condition for a line $Ax + By + C = 0$ to be tangent to a circle $x^2 + y^2 = r^2$ is:

$$
\text{Distance from the center to the line} = \text{radius}
$$

Since the circle is centered at (0, 0), the distance d from the center to the line is:

$$
d = \frac{|C|}{\sqrt{A^2 + B^2}}
$$

For tangency:

$$
d = r
$$

Note that because the line is tangent at (5, 4), the point (5, 4) lies on both the circle and the line.

Step 2: Establishing the Known Point on the Line

Given point: (5, 4)

Substituting into the line's equation:

$$
10(5) + P(4) = Q
$$
$$
50 + 4P = Q
$$

This provides one relationship between P and Q:

$$
Q = 50 + 4P
$$

Step 3: Calculating the Radius

Since the line is tangent at (5, 4), and this point lies on the circle, the radius r can be obtained by calculating the distance from the center (0,0) to the point (5,4):

$$
r = \sqrt{(5)^2 + (4)^2} = \sqrt{25 + 16} = \sqrt{41}
$$

Step 4: Expressing the Distance from the Center to the Line

Recall the formula for the distance from (0, 0) to the line:

$$
d = \frac{|C|}{\sqrt{A^2 + B^2}} = \frac{|\,-Q\,|}{\sqrt{10^2 + P^2}} = \frac{|Q|}{\sqrt{100 + P^2}}
$$

For the line to be tangent to the circle, this distance must equal the radius:

$$
\frac{|Q|}{\sqrt{100 + P^2}} = \sqrt{41}
$$

Since the point (5, 4) lies on the line, and substituting $Q = 50 + 4P$, we get:

$$
\frac{|50 + 4P|}{\sqrt{100 + P^2}} = \sqrt{41}
$$

Step 5: Solving for P

Squaring both sides:

$$
\frac{(50 + 4P)^2}{100 + P^2} = 41
$$

Cross-multiplied:

$$
(50 + 4P)^2 = 41(100 + P^2)
$$

Expanding the numerator:

$$
(50)^2 + 2 \times 50 \times 4P + (4P)^2 = 41 \times 100 + 41 P^2
$$

$$
2500 + 400P + 16 P^2 = 4100 + 41 P^2
$$

Bring all terms to one side:

$$
2500 + 400P + 16 P^2 - 4100 - 41 P^2 = 0
$$

Simplify:

$$
(2500 - 4100) + 400P + (16 P^2 - 41 P^2) = 0
$$

$$
-1600 + 400P - 25 P^2 = 0
$$

Divide through by 25 to simplify:

$$
-64 + 16 P - P^2 = 0
$$

Rearranged as standard quadratic:

$$
-P^2 + 16 P - 64 = 0
$$

Multiply through by -1:

$$
P^2 - 16 P + 64 = 0
$$

Step 6: Solving the Quadratic Equation

Apply quadratic formula:

$$
P = \frac{16 \pm \sqrt{(16)^2 - 4 \times 1 \times 64}}{2}
$$

Calculate discriminant:

$$
\Delta = 256 - 256 = 0
$$

Since discriminant is zero, there is one unique solution:

$$
P = \frac{16}{2} = 8
$$

Step 7: Finding Q

Recall:

$$
Q = 50 + 4 P
$$

Substitute $P = 8$:

$$
Q = 50 + 4 \times 8 = 50 + 32 = 82
$$

Final Result:

The value of P is 8, and Q is 82.

Deep Dive: Geometric Significance and Validation

Validating the Tangency

- The line: $10x + 8 y = 82$
- The distance from (0, 0) to the line:

$$
d = \frac{|82|}{\sqrt{100 + 64}} = \frac{82}{\sqrt{164}} \approx \frac{82}{12.806} \approx 6.4
$$

- Radius: $r = \sqrt{41} \approx 6.4$

- The point (5, 4) lies on the line:

$$
10 \times 5 + 8 \times 4 = 50 + 32 = 82
$$

which confirms the point is on the line.

- The point (5, 4) is at a distance:

$$
\sqrt{(5)^2 + (4)^2} = \sqrt{41}
$$

from the origin, which matches the radius, confirming that the line is tangent at (5, 4).

Geometric Consistency

This validation consolidates that the solution aligns with the geometric conditions for tangency, confirming the accuracy of the derived values.

Broader Implications and Applications

This problem exemplifies fundamental concepts in coordinate geometry, especially the relationship between lines and circles. Applications extend to various fields including:

- Engineering: Designing tangent contact points in mechanical linkages.
- Physics: Analyzing trajectories tangent to circular fields.
- Computer Graphics: Rendering tangent lines and circles with precision.
- Robotics: Path planning with obstacle avoidance involving circles and tangents.

It also demonstrates the power of algebraic manipulation to solve geometric constraints efficiently.

Conclusion

Through a systematic approach combining algebra, geometry, and logical reasoning, we have determined that:

- $P = 8$
- $Q = 82$

satisfy the conditions where the line $10x + Py = Q$ is tangent at point (5, 4) to a circle centered at the origin with radius $\sqrt{41}$.

This investigation not only solves the posed problem but also underscores the elegance of coordinate geometry in resolving complex geometric configurations. Such problems reinforce the importance of foundational principles, meticulous calculations, and validation in mathematical problem solving.

Key Takeaways:

- The tangent condition relates the perpendicular distance from the circle's center to the line.
- Substituting known points allows establishing relationships between line parameters.
- Quadratic equations emerge naturally in geometric problems, often requiring careful algebraic manipulation.
- Validating solutions through substitution ensures consistency with geometric intuition.

This detailed exploration underscores

The Line 10x Py Q Is A Tangent At The Point 5 4 In Another Circle With Centre 0 0 Find The Value

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