# The Random Variable N Takes Non-negative Integer Values. Show That E(N)= K=0[infinity]P(N>k) Provided

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Understanding the behavior and properties of random variables is fundamental in probability theory and statistics. One particularly important class of random variables are those that take on non-negative integer values, typically denoted as N. These are often used to model discrete events such as the number of arrivals at a queue, the number of successes in a sequence of Bernoulli trials, or the count of occurrences of an event within a fixed period. A key property related to such variables is the relationship between their expected value and tail probabilities, specifically expressed as:

$$ E(N) = \sum_{k=0}^\infty P(N > k) $$

This formula elegantly links the expected value of N to the sum of its tail probabilities, providing both a theoretical insight and a practical computational approach. This article explores this relationship in depth, demonstrating why it holds and how it can be employed in various probabilistic analyses.

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## Understanding Non-negative Integer-Valued Random Variables

### Definition and Basic Properties

A random variable $N$ is said to take non-negative integer values if:

- $N \in \{0, 1, 2, 3, \dots\}$
- The probability $P(N = k) \geq 0$ for all $k \geq 0$
- The sum of probabilities over all possible values sums to 1:

$$
\sum_{k=0}^\infty P(N = k) = 1
$$

This class of variables is discrete, and their distribution can be characterized fully by the probability mass function (pmf), $p(k) = P(N=k)$.

### Examples of Non-negative Integer-Valued Variables

Some common examples include:

- Poisson distribution: Counts of events occurring randomly over a fixed interval.
- Geometric distribution: Number of trials until the first success.
- Binomial distribution: Number of successes in a fixed number of independent Bernoulli trials.
- Negative binomial distribution: Number of trials needed to achieve a certain number of successes.

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## Expected Value of a Non-negative Integer-Valued Random Variable

### Definition of Expectation

The expected value (or mean) of a discrete random variable $N$ is defined as:

$$
E(N) = \sum_{k=0}^\infty k \cdot P(N=k)
$$

This summarizes the 'average' or 'long-term' value of the random variable over many trials or observations.

### Intuitive Understanding of Expectation

- It can be viewed as a weighted average.
- The weights are the probabilities associated with each possible value.
- For non-negative integers, the expectation provides a measure of the typical size of $N$.

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## Expressing Expectation in Terms of Tail Probabilities

## The Core Relationship

The key result we are examining states that for a non-negative integer-valued random variable $N$:

$$
E(N) = \sum_{k=0}^\infty P(N > k)
$$

This expresses the expectation as an infinite sum over tail probabilities, which are the probabilities that $N$ exceeds a certain value $k$.

## Why Is This Relationship Important?

- It offers an alternative way to compute $E(N)$ without directly summing over all $k$ with weights $k$.
- It provides insights into the tail behavior of $N$.
- It simplifies calculations in some cases, especially when tail probabilities are easier to evaluate or estimate.

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# Proving the Relationship: $E(N) = \sum_{k=0}^\infty P(N > k)$

## Preliminaries and Assumptions

To establish the relationship, we assume:

- $N$ is a non-negative integer-valued random variable.
- The sum $\sum_{k=0}^\infty P(N > k)$ converges (which is generally true when $E(N)$ is finite).

## Step-by-Step Proof

1. **Express $P(N > k)$ in terms of probability mass function:**
   $$
   P(N > k) = \sum_{j=k+1}^\infty P(N = j)
   $$

2. **Rewrite the expectation as a sum:**
   $$
   E(N) = \sum_{k=0}^\infty k \cdot P(N=k)

\]

3. **Relate \( k \) to tail probabilities:**
   Notice that for each \( k \geq 0 \),
   \[
   P(N > k) = \sum_{j=k+1}^\infty P(N = j)
   \]
   which can be viewed as the sum of probabilities of all \( j \) greater than \( k \).

4. **Express \( E(N) \) as a sum over tail probabilities:**
   Observe that
   \[
   \sum_{k=0}^\infty P(N > k) = \sum_{k=0}^\infty \sum_{j=k+1}^\infty P(N = j)
   \]
   Swap the order of summation:
   \[
   = \sum_{j=1}^\infty P(N = j) \sum_{k=0}^{j-1} 1
   \]
   Since \( \sum_{k=0}^{j-1} 1 = j \), this simplifies to:
   \[
   \sum_{j=1}^\infty j \cdot P(N = j)
   \]
   which is exactly \( E(N) \).

5. **Conclusion:** Therefore,
   \[
   E(N) = \sum_{k=0}^\infty P(N > k)
   \]
   as desired.

## Summary of the Proof

- The key idea hinges on swapping the order of summation.
- The tail probabilities encapsulate the cumulative probability of \( N \) exceeding certain thresholds.
- The proof assumes convergence, which is guaranteed when \( E(N) \) is finite.

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# Implications and Applications of the Relationship

## Practical Computation of $E(N)$

- When the pmf is complicated, but tail probabilities are accessible, the sum provides an alternative computation method.
- For certain distributions, tail probabilities have closed forms, simplifying the calculation.

## Analysis of Tail Behavior

- The sum over tail probabilities offers insights into the distribution's tail decay.
- Fast decay of $P(N > k)$ implies finite expectation and possibly finite higher moments.

## Bounding the Expectation

- Inequalities involving tail probabilities can be used to bound the expected value.
- For example, Markov's inequality relates tail probabilities to moments.

## Extensions to Other Moments

- Similar techniques can be used to analyze higher moments, such as variance, by involving tail probabilities.

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# Examples Demonstrating the Relationship

## Poisson Distribution

- For $N \sim \text{Poisson}(\lambda)$, the tail probability $P(N > k)$ can be expressed via the incomplete gamma function or summed directly.
- Expectation: $E(N) = \lambda$.
- Sum of tail probabilities matches $\lambda$, illustrating the relationship.

### Geometric Distribution

- For $N \sim \text{Geometric}(p)$, the tail probability:

$$
P(N > k) = (1 - p)^{k+1}
$$

- Sum over $k$:

$$
\sum_{k=0}^\infty (1 - p)^{k+1} = \frac{1 - p}{p} = E(N)
$$

- Confirming the theoretical result.

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## Limitations and Conditions

### Finite Expectation Requirement

- The relationship holds when $E(N)$ is finite.
- If $E(N)$ is infinite, the sum of tail probabilities diverges.

### Convergence Considerations

- The sum $\sum_{k=0}^\infty P(N > k)$ must converge.
- Usually guaranteed when the tail probabilities decay sufficiently fast.

### Non-negative Assumption

- The derivation relies on $N$ being non-negative.
- For variables taking negative values, the relationship does not hold as is.

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## Conclusion

The relationship

$$
E(N) = \sum_{k=0}^\infty P(N > k)
$$

provides a powerful and elegant connection between the expected value of a non-negative integer-valued random variable and its tail probabilities. This link not only simplifies computations in many scenarios but also

# Frequently Asked Questions

## What does it mean for a random variable N to take non-negative integer values?

It means that N can only take values in the set {0, 1, 2, 3, ...}, i.e., it is a discrete variable with non-negative integers as its possible outcomes.

## How is the expectation E(N) of a non-negative integer-valued random variable N expressed in terms of tail probabilities?

For such variables, E(N) can be written as the sum over all k from 0 to infinity of P(N > k), i.e., $E(N) = \sum_{k=0}^\infty P(N > k)$.

## Why does the sum over tail probabilities P(N > k) equal the expected value E(N)?

This relationship holds because the expectation of a non-negative integer-valued random variable can be viewed as the area under its tail probability curve, effectively summing the probabilities of exceeding each integer k.

## Can you provide a brief proof or intuition for why $E(N) = \sum_{k=0}^\infty P(N > k)$?

Yes; since N takes values in non-negative integers, $E(N) = \sum_{n=0}^\infty n P(N = n)$. Recognizing that $n = \sum_{k=0}^{n-1} 1$, we interchange sums to get $E(N) = \sum_{k=0}^\infty P(N > k)$, because P(N > k) sums the probabilities where N exceeds each k.

## What assumptions are necessary for the formula $E(N) = \sum_{k=0}^\infty P(N > k)$ to hold?

The key assumption is that N is a non-negative, discrete random variable with a finite or infinite expectation, ensuring the sum converges. Typically, N should be non-negative and integrable (i.e., $E(N) < \infty$).

## How is this tail sum representation useful in practice?

It allows calculating the expectation of N using tail probabilities, which can be easier to estimate or bound, especially in cases like geometric or Poisson distributions where tail probabilities are well-understood.

### Is the formula E(N) = ∑_{k=0}^∞ P(N > k) valid for continuous random variables?

No, this formula specifically applies to discrete, non-negative integer-valued variables. For continuous variables, the expectation is expressed as an integral involving the survival function: E(N) = ∫_{0}^∞ P(N > t) dt.

## Additional Resources

Random Variable N and Its Expectation: Exploring the Relationship Between E(N) and P(N > k)

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## Introduction: Unveiling the Significance of the Random Variable N

In the realm of probability theory, the concept of a random variable forms the backbone of modeling uncertainty across diverse fields such as statistics, engineering, finance, and computer science. Among the myriad types of random variables, discrete random variables that take non-negative integer values hold particular importance, especially when analyzing counts, waiting times, or the number of occurrences of an event.

Imagine observing a process that results in a count—say, the number of emails received in a day, the number of failures before success in a sequence of trials, or the number of customers arriving at a store. These are naturally modeled by a discrete random variable $N$ taking values in $\{0, 1, 2, \ldots \}$.

The crux of understanding such variables lies in their expectation, $E(N)$, which provides a measure of the "average" or "expected" value of the count over many repetitions. An elegant and powerful result links this expectation directly to the tail probabilities $P(N > k)$, offering an alternative perspective that is both insightful and practical. This relationship states:

$$
E(N) = \sum_{k=0}^{\infty} P(N > k)
$$

This article aims to explore this relationship in depth, elucidate its proof, interpret its significance, and demonstrate how it serves as a fundamental tool in probability theory.

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# Understanding the Expectation of a Non-negative Integer-Valued Random Variable

## Defining the Random Variable N

Let $N$ be a random variable such that:

$$
N: \Omega \to \{0, 1, 2, \ldots\}
$$

where $\Omega$ is the sample space. The assumption that $N$ takes non-negative integer values aligns with numerous real-world phenomena modeled as counting processes.

Key properties:

- Non-negativity: $N \geq 0$ almost surely.
- Discrete distribution: $P(N = k)$ for each $k \in \mathbb{N}_0$.
- Probability mass function (pmf): $p(k) = P(N = k)$.

The expectation of $N$, denoted $E(N)$, is formally defined as:

$$
E(N) = \sum_{k=0}^\infty k \cdot p(k)
$$

assuming the sum converges, which it does for non-negative variables if $E(N)$ is finite.

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# The Fundamental Relationship: Connecting E(N) with Tail Probabilities

## Statement of the Result

The essential identity we explore is:

$$
\boxed{
E(N) = \sum_{k=0}^\infty P(N > k)
}
$$

\]

This powerful formula allows us to compute or estimate the expectation of \( N \) solely based on its tail probabilities, often simplifying calculations, especially in complex distributions or when tail bounds are easier to obtain.

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## Intuitive Understanding

- For each value \( k \), the probability \( P(N > k) \) indicates the likelihood that \( N \) exceeds \( k \).
- Summing these tail probabilities over all \( k \geq 0 \) effectively aggregates all "exceedance" probabilities, capturing the total expected count.
- This perspective shifts focus from directly summing over probabilities \( p(k) \) weighted by \( k \), to summing tail probabilities, which can sometimes be more accessible or intuitive.

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# Step-by-Step Derivation and Proof of the Identity

## Preliminary Observations

Given that \( N \) is non-negative and integer-valued, the expectation can be written as:

\[
E(N) = \sum_{k=0}^\infty k \cdot p(k)
\]

Our goal is to transform this sum into an expression involving \( P(N > k) \).

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## Rearrangement via Indicator Functions

Recall that the indicator function \( \mathbf{1}_{\{N > k\}} \) equals 1 if \( N > k \), and 0 otherwise. Then,

$$
N = \sum_{k=0}^\infty \mathbf{1}_{\{N > k\}}
$$

since for a fixed $N$:

- When $N = 0$, all indicators are zero, sum is zero.
- When $N = n \geq 1$, then:

$$
\sum_{k=0}^\infty \mathbf{1}_{\{n > k\}} = n
$$

because exactly $n$ indicators are 1 (for $k = 0, 1, ..., n-1$), and the rest are zero.

Therefore:

$$
N = \sum_{k=0}^\infty \mathbf{1}_{\{N > k\}}
$$

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## Taking Expectations on Both Sides

Since the sum involves non-negative random variables, Fubini's theorem allows us to interchange the expectation and the summation:

$$
E(N) = E \left[ \sum_{k=0}^\infty \mathbf{1}_{\{N > k\}} \right] = \sum_{k=0}^\infty E [ \mathbf{1}_{\{N > k\}} ]
$$

But,

$$
E [ \mathbf{1}_{\{N > k\}} ] = P(N > k)
$$

because the expectation of an indicator function is its probability.

Putting it together:

$$
\boxed{
E(N) = \sum_{k=0}^\infty P(N > k)
}
$$

This completes the proof, confirming the fundamental relationship between the expectation of $N$ and its tail probabilities.

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# Implications and Applications of the Result

## 1. Simplification in Calculations

In many cases, directly computing $E(N)$ from the pmf $p(k)$ can be cumbersome, especially if the pmf involves complex expressions or is unknown explicitly. However, tail probabilities, $P(N > k)$, might be easier to estimate or bound, especially if tail bounds or cumulative distribution functions are known.

Example:

- For a geometric distribution $N \sim \text{Geom}(p)$, the tail probability:

$$
P(N > k) = (1 - p)^{k+1}
$$

is straightforward, enabling direct summation to find $E(N)$.

---

## 2. Bounding Expectations via Tail Bounds

The identity allows for deriving bounds on $E(N)$:

$$
E(N) = \sum_{k=0}^\infty P(N > k) \leq \sum_{k=0}^\infty \text{(some upper bound on } P(N > k))
$$

which can be useful in probabilistic inequalities and concentration bounds.

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## 3. Intuitive Interpretation in Queueing and

## Reliability

- In queueing systems, $N$ might represent the number of customers in the system.
- The tail probability $P(N > k)$ indicates the probability of having more than $k$ customers.
- Summing these tail probabilities gives the average number of customers, providing insights into system capacity and performance.

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# Examples Demonstrating the Relationship

## Example 1: Geometric Distribution

Let $N \sim \text{Geometric}(p)$, with:

$$
P(N = k) = p (1 - p)^k, \quad k = 0, 1, 2, \ldots
$$

The tail probability:

$$
P(N > k) = (1 - p)^{k+1}
$$

Using the identity:

$$
E(N) = \sum_{k=0}^\infty P(N > k) = \sum_{k=0}^\infty (1 - p)^{k+1} = (1 - p) \sum_{k=0}^\infty (1 - p)^k
$$

which sums to:

$$
(1 - p) \cdot \frac{1}{1 - (1 - p)} = (1 - p) \cdot \frac{1}{p} = \frac{1 - p}{p}
$$

This matches the known expectation:

$$
E(N) = \frac{1 - p}{p}
$$

This example illustrates how the tail sum directly yields the expectation.

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## Example 2: Poisson Distribution

For a Poisson random variable $N \sim \text{Poisson}(\lambda)$, the tail probability:

$$
P(N > k) = 1 - P(N \leq k) = 1 - \sum_{i=0}^k \frac{e^{-\lambda} \lambda^i}{i!}
$$

While the sum for expectation involves $E(N) = \lambda$, the

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