Two Sides And An Angle Are Given Below. Determine Whether The Given Information Results In One Triangle,

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Understanding the conditions that determine the existence and uniqueness of a triangle is fundamental in geometry. When given two sides and an angle, mathematicians analyze whether these measurements can form one, multiple, or no triangles at all. This problem is a classic in triangle geometry and is often encountered in various applications, including engineering, architecture, and navigation. The core question revolves around interpreting the given data correctly and applying the appropriate theorems or rules, such as the Law of Sines, Law of Cosines, and the SSA (Side-Side-Angle) condition, to arrive at a conclusion.

In this comprehensive guide, we will explore how to determine whether a set of two sides and an angle produces exactly one triangle, multiple triangles, or no triangle at all. We will delve into the different cases, the geometric principles involved, and provide step-by-step methods to analyze each scenario. By the end of this article, you will be equipped with the knowledge to approach such problems confidently and accurately.

Understanding the Basic Concepts

Before diving into the problem-solving techniques, it is essential to understand some fundamental concepts related to triangles and the given data.

Types of Data Given

When given two sides and an angle, the data can be categorized as:

- Two sides and the included angle (SAS): The angle is between the two sides.
- Two sides and a non-included angle (SSA): The angle is not between the two sides; it is adjacent to one of the given sides.

The distinction between SAS and SSA is crucial because the methods to analyze the problem differ significantly.

Triangle Construction Principles

- A triangle is uniquely determined when given SAS data.
- When given ASA or SSS data, the triangle is also uniquely determined.
- SSA (Two sides and a non-included angle) can produce zero, one, or two triangles, making it a ambiguous case.

Analyzing the Given Data: Two Sides and an Angle

The core analysis depends on the position of the given angle relative to the sides, which leads to different cases.

Case 1: The Given Angle Is Included Between the Two Sides (SAS)

- Definition: The known angle is between the two given sides.
- Implication: The triangle is uniquely determined.
- Reasoning: The Law of Cosines can be used to find the third side, and then the triangle is fully determined.
- Result: Exactly one triangle can be constructed.

Case 2: The Given Angle Is Adjacent to One of the Known Sides (SSA or Non-Included Angle)

- Definition: The known angle is not between the two given sides; it is adjacent to one of them.
- Implications: This case can produce zero, one, or two triangles.
- Reasoning: The problem is ambiguous because the given data do not guarantee a unique solution.

Determining the Number of Triangles in SSA (Side-Side-Angle) Cases

The SSA configuration is notorious for ambiguity. To analyze whether one, two, or no triangles can be formed, follow these steps.

Step-by-Step Approach for SSA Data

1. Identify the given data:

- Known side a
- Known angle A
- The side b adjacent to angle A

2. Determine the possible scenarios:
- Is side a opposite the given angle A?
- Is side b the side adjacent to the angle?

3. Apply the Law of Sines:

$$
\frac{a}{\sin A} = \frac{b}{\sin B}
$$

4. Calculate $\sin B$:

$$
\sin B = \frac{b \sin A}{a}
$$

5. Analyze $\sin B$:
- If $\sin B > 1$: No solution (no triangle).
- If $\sin B = 1$: One solution (right triangle).
- If $0 < \sin B < 1$: Two solutions possible (two triangles).

6. Check for the second solution:

$$
B' = 180^\circ - B
$$

If B' is valid (i.e., the sum of angles less than 180°), then two triangles are possible.

Applying the Law of Cosines and Law of Sines

The Law of Cosines and Law of Sines are vital tools in solving triangle problems:

- Law of Cosines:

$$
c^2 = a^2 + b^2 - 2ab \cos C
$$

- Law of Sines:

\[

$$
\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}
$$

Depending on the given data's configuration, these laws help find missing sides or angles and determine whether a triangle exists and if it is unique.

Practical Examples and Visualizations

Let's analyze some typical examples to illustrate how to determine the number of triangles from given two sides and an angle.

Example 1: SAS Case

Given:

- $AB = 7\, \text{cm}$
- $AC = 10\, \text{cm}$
- $\angle BAC = 60^\circ$

Analysis:

Since the angle is between the two sides, we apply the Law of Cosines:

$$
BC^2 = AB^2 + AC^2 - 2 \times AB \times AC \times \cos 60^\circ
$$

$$
BC^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times 0.5
$$

$$
BC^2 = 49 + 100 - 70 = 79
$$

$$
BC = \sqrt{79} \approx 8.89\, \text{cm}
$$

Conclusion: Only one triangle can be formed because SAS guarantees a unique solution.

Example 2: SSA Case with Two Possible Triangles

Given:

- $\angle A = 30^\circ$
- $a = 8\, \text{cm}$
- $b = 10\, \text{cm}$

Analysis:

Calculate $\sin B$:

$$
\sin B = \frac{b \sin A}{a} = \frac{10 \times \sin 30^\circ}{8} = \frac{10 \times 0.5}{8} = \frac{5}{8} = 0.625
$$

Since $0 < 0.625 < 1$, two solutions for B:

- $B = \arcsin 0.625 \approx 38.68^\circ$
- $B' = 180^\circ - 38.68^\circ \approx 141.32^\circ$

Check the sum of angles:

- For $B = 38.68^\circ$:

$$
A + B + C = 180^\circ \Rightarrow C = 180^\circ - (30^\circ + 38.68^\circ) = 111.32^\circ
$$

- For $B' = 141.32^\circ$:

$$
C = 180^\circ - (30^\circ + 141.32^\circ) = 8.68^\circ
$$

Both sets of angles are valid, so two triangles can be constructed.

Conclusion: SSA data here leads to two possible triangles.

Summary of Conditions for One, Multiple, or No Triangle

Given Data Type	Conditions	Number of Triangles	Explanation

SAS (Side-Angle-Side)	Included angle between two sides	Exactly one	The Law of Cosines guarantees a unique solution.
ASA or SSS	Known angles or sides fully determine the triangle	Exactly one	These are sufficient for a unique solution.
SSA (Side-Side-Angle)	Non-included angle, ambiguous case	Zero, One, or Two	Depends on the specific measurements and the sine rule analysis.

Additional Tips for Solving Triangle Problems

- Always identify whether the given angle is included or non-included.
- Use the Law of Cosines for SAS cases.
- Use the Law of Sines for SSA and ASA cases.
- Check the validity of solutions by verifying the sum of angles and the triangle inequality.
- Be cautious with the ambiguous SSA case; perform calculations for both possible solutions when applicable.
- Draw accurate diagrams to visualize the problem, which helps in understanding the possible configurations.

Conclusion

Determining whether the given information results in one triangle requires careful analysis of the data and understanding of the underlying principles. The key lies in recognizing the

Frequently Asked Questions

If two sides and the included angle are given in a triangle, can we uniquely determine the triangle?

Yes, if two sides and the included angle are given (SAS condition), the triangle is uniquely determined.

Given two sides and a non-included angle, is it always possible to form a triangle?

Not necessarily. This is the SSA configuration, which can lead to zero, one, or two possible triangles, so it doesn't always guarantee a unique triangle.

Does knowing two sides and an angle opposite one of the sides guarantee a unique triangle?

No, because this is the SSA case, which may result in no triangle, one triangle, or two triangles, so it does not always guarantee a unique solution.

When given two sides and an angle, what criterion ensures the formation of exactly one triangle?

If the given angle is included between the two sides (SAS), then exactly one triangle is formed.

What is the key difference between the SAS and SSA cases regarding triangle formation?

SAS guarantees a unique triangle, whereas SSA can lead to multiple, one, or no triangles, making it less reliable for unique determination.

How can one determine whether the given two sides and an angle result in one triangle?

By applying the Law of Sines or Law of Cosines and checking the possible solutions, you can determine if the given data yields exactly one triangle.

Additional Resources

Two Sides And An Angle Are Given Below. Determine Whether The Given Information Results In One Triangle—this is a common problem scenario in triangle geometry that tests your understanding of triangle properties, the Law of Sines, Law of Cosines, and the criteria for triangle existence. Whether you're a student preparing for exams or a math enthusiast exploring geometric principles, grasping how given elements influence the number of possible triangles is fundamental. In this guide, we'll explore the key concepts, step-by-step methods, and decision-making strategies to determine whether a set of given information results in a unique triangle, multiple triangles, or no triangle at all.

Understanding the Basics of Triangle Construction

Before delving into specific cases, it's essential to understand the foundational principles governing triangles:

- Triangle Inequality Theorem: For any triangle, the sum of the lengths of any two sides must be greater than the third side.
- Given Elements: When solving triangle problems, the given data typically include sides, angles, or a combination of both.
- Ambiguous Case (SSA): A common scenario where two sides and a non-included angle are given, which can lead to zero, one, or two possible triangles.

The Core Question: What Does "Two Sides And An Angle Are Given Below" Mean?

When a problem states "Two sides and an angle are given below," it usually refers to one of the following configurations:

1. Two sides and the included angle (SAS): The angle is between the two given sides.
2. Two sides and a non-included angle (SSA): The angle is adjacent to only one of the given sides, not between them.

The distinction between these configurations is crucial because:

- SAS always results in exactly one triangle (assuming the data is consistent).
- SSA can result in zero, one, or two triangles, depending on the specific measurements.

Case 1: Two Sides and the Included Angle (SAS)

Definition

In SAS, you are given:

- Two sides, say, AB and AC.
- The included angle between these sides, ∠A.

Key Features

- The triangle is uniquely determined if the given data is consistent.
- The Law of Cosines is typically used to find the third side.

Step-by-Step Approach

1. Identify the given data:

- Sides: $b = AB$, $c = AC$
- Included angle: A

2. Calculate the third side:

- Use Law of Cosines:

$$
BC^2 = AB^2 + AC^2 - 2 \times AB \times AC \times \cos A
$$

- Determine whether the computed length is valid (positive and satisfies triangle inequality).

3. Find remaining angles:

- Use Law of Sines:

$$
\frac{\sin B}{b} = \frac{\sin C}{c} = \frac{a}{\text{side opposite }A}
$$

- Or, use Law of Cosines to find other angles if needed.

4. Conclusion:

- Since SAS defines a specific triangle, exactly one solution exists (unless the data is inconsistent).

Case 2: Two Sides and a Non-Included Angle (SSA)

Definition

In SSA, the given data are:

- Two sides: a and b.
- An angle: $\angle A$, which is not between sides a and b.

Why Is SSA Special?

The SSA configuration is known as the ambiguous case because it can produce:

- No triangle if the data is inconsistent.
- Exactly one triangle if the data corresponds to a unique configuration.
- Two triangles if the data allows for two different valid triangles.

Visual Understanding

Imagine side a opposite angle A, and side b opposite angle B. The given angle A is adjacent to side a, but side b is somewhere else in the figure.

Determining the Number of Triangles in SSA

Step 1: Identify the knowns

Suppose you're given:

- $\angle A$
- Sides a and b

Step 2: Calculate the height

- Drop a perpendicular from the vertex of $\angle A$ to side BC, creating height h:

$$
h = b \times \sin A
$$

Step 3: Analyze the possible cases

Based on the length a relative to h and b, determine the number of solutions:

1. No solution:

- If $a < h$, then side a is too short to reach the line BC, meaning no triangle exists.

2. One solution:

- If $a = h$, then the triangle is right-angled, and only one triangle is possible.

3. Two solutions:

- If $a > h$, and $a < b$, then two triangles are possible because the side a can be "flipped" over the height, creating two different configurations.

Step 4: Formal criteria

Using the Law of Sines, determine if a second triangle exists:

$$
\sin B = \frac{b \times \sin A}{a}
$$

- If $\sin B \leq 1$, then:

- One triangle if $\sin B = 1$ (right angle).
- Two triangles if $\sin B < 1$ and the corresponding angles are acute or obtuse, leading to two different possible triangles.

- If $\sin B > 1$, then no triangle exists.

Practical Examples and Decision Matrix

Example 1: SAS Scenario

Given:

- $AB = 7$
- $AC = 10$
- $\angle A = 60^\circ$

Analysis:

- Use Law of Cosines to find side BC:

$$
BC^2 = 7^2 + 10^2 - 2 \times 7 \times 10 \times \cos 60^\circ = 49 + 100 - 2 \times 7 \times 10 \times 0.5
$$

$$
BC^2 = 149 - 70 = 79
$$

$$
BC \approx \sqrt{79} \approx 8.89
$$

- Since the data is consistent, exactly one triangle exists.

Example 2: SSA Scenario

Given:

- $\angle A = 40^\circ$
- $a = 8$
- $b = 10$

Analysis:

- Calculate $h = b \times \sin A = 10 \times \sin 40^\circ \approx 10 \times 0.6428 \approx 6.43$

- Since $a = 8 > h$:

- Check if $a < b$:

$$
8 < 10 \quad \text{(Yes)}
$$

- So, two triangles are possible.

- Calculate $\sin B$:

$$
\sin B = \frac{b \times \sin A}{a} = \frac{10 \times 0.6428}{8} \approx 0.8035
$$

- Since $\sin B < 1$, two solutions exist:

- $B_1 = \arcsin 0.8035 \approx 53.1^\circ$
- $B_2 = 180^\circ - 53.1^\circ \approx 126.9^\circ$

- Corresponding C angles:

$$
C_1 = 180^\circ - A - B_1 \approx 87.9^\circ
$$

$$
C_2 = 180^\circ - A - B_2 \approx 13.1^\circ
$$

- Thus, two different triangles are possible.

Summary Table: Triangle Possibilities Based on Given Data

Scenario	Conditions	Number of Triangles	Notes
SAS (two sides and included angle)	Data consistent	Exactly 1	Law of Cosines applies directly
SSA (two sides and non-included angle)	$a < h$	0	No triangle possible
SSA	$a = h$	1	Right triangle, unique solution
SSA	$h < a < b$	2	Two possible triangles
SSA	$a \geq b$	1	One triangle (since the second would violate triangle inequality)

Final Recommendations for Problem Solving

- Identify the given elements: sides, angles, and their positions.
- Determine the configuration: SAS, SSS, ASA, AAS, or SSA.
- Apply appropriate theorems:

- Law of Cosines for SAS and

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