Use The Elimination Method To Find A General Solution For The Given Linear System, Where Differentiation

Use The Elimination Method To Find A General Solution For The Given Linear System, Where Differentiation

When tackling systems of differential equations, particularly those that are linear and involve multiple variables, finding a general solution can seem complex. However, the elimination method offers a systematic approach to simplify such systems, allowing you to derive a unified solution. This technique, combined with differentiation, helps in reducing the system to a single higher-order differential equation, which can then be solved using standard methods. In this comprehensive guide, we'll explore how to effectively apply the elimination method to linear systems involving differentiation, step-by-step, to find their general solutions.

Understanding the Elimination Method in Differential Equations

Before diving into the process, it’s essential to understand what the elimination method entails and why it is useful in the context of differential equations.

What is the Elimination Method?

The elimination method is a technique traditionally used to solve systems of algebraic equations. When extended to differential equations, it involves:

- Combining the equations to eliminate one of the variables
- Deriving a higher-order differential equation involving only one variable
- Solving this simplified equation to find the general solution

This method leverages the relationships between the equations, exploiting their linearity to systematically eliminate variables.

Why Use the Elimination Method?

The main advantages of using this method include:

- Simplification of complex systems into a single differential equation
- Clearer pathway to finding the general solution
- Applicability to linear systems with constant coefficients
- Facilitating the use of standard solution techniques for higher-order differential equations

Step-by-Step Procedure for Applying the Elimination Method

Applying the elimination method to a given linear system involving derivatives involves several methodical steps. Here’s a structured outline:

1. Write Down the System Clearly

Typically, the system will consist of two or more first-order linear differential equations, such as:

Equation 1: $a_1(x)y' + b_1(x)z' + c_1(x)y + d_1(x)z = e_1(x)$
Equation 2: $a_2(x)y' + b_2(x)z' + c_2(x)y + d_2(x)z = e_2(x)$

Ensure all equations are explicitly written, with derivatives clearly indicated.

2. Identify the Variables and Their Derivatives

Determine which variables and derivatives are involved. Usually, in systems involving two variables $y(x)$ and $z(x)$, their derivatives y' and z' are present.

3. Differentiate Equations if Necessary

Sometimes, to facilitate elimination, you might need to differentiate one of the equations with respect to x to create expressions that can be combined with the other equations.

4. Manipulate Equations to Eliminate a Variable

Use algebraic operations to:

- Multiply equations by suitable functions to align coefficients
- Subtract or add equations to cancel out one variable's derivative

For example:

- Multiply Equation 1 by coefficient A
- Multiply Equation 2 by coefficient B
- Subtract one from the other to eliminate y' or z'

This yields a new equation involving only one variable and its derivatives.

5. Derive a Higher-Order Differential Equation

Continue manipulating the equations until you obtain a single differential equation involving only one variable (say, y), which may be of second order or higher.

6. Solve the Resulting Differential Equation

Apply standard techniques for solving linear differential equations with constant or variable coefficients:

- Characteristic equation method for constant coefficients
- Method of undetermined coefficients
- Variation of parameters
- Reduction of order for second-order equations

7. Back-Substitute to Find Remaining Variables

Once the primary variable's general solution is known, substitute back into original equations to find the other variables.

Application Examples

To clarify this process, let's explore an example involving a system of two linear differential equations.

Example System:

```
Equation 1: y' + y + z = 0
Equation 2: z' + 2y = 0
```

Objective: Find the general solution for y(x) and z(x).

Applying the Elimination Method:

Step 1: Differentiate Equation 2:

```
z' + 2y = 0
Differentiating both sides:
z'' + 2y' = 0
```

Step 2: Express y' from Equation 1:

$$y' = -y - z$$

Step 3: Substitute y' into the differentiated Equation 2:

$$z'' + 2(-y - z) = 0$$
$$\Rightarrow z'' - 2y - 2z = 0$$

Step 4: Express y in terms of z and z' using Equation 2:

From Equation 2:

$$z' + 2y = 0$$
$$\Rightarrow y = -z'/2$$

Step 5: Replace y in the above expression into the previous equation:

$$z'' - 2(-z'/2) - 2z = 0$$
$$\Rightarrow z'' + z' - 2z = 0$$

This is a second-order linear differential equation in z(x).

Step 6: Solve the differential equation:

Characteristic equation:

$$r^2 + r - 2 = 0$$
$$\Rightarrow r = [-1 \pm \sqrt{(1 + 8)}]/2 = [-1 \pm 3]/2$$

Solutions:

- r = 1
- r = -2

General solution for z(x):

$$z(x) = C_1 e^{x} + C_2 e^{-2x}$$

Step 7: Find y(x):

Recall that $y = -z'/2$

Calculate z':

$$z' = C_1 e^{x} - 2 C_2 e^{-2x}$$

Thus:

$$y(x) = -(z'/2) = -(C_1 e^{x} - 2 C_2 e^{-2x})/2 = -(C_1/2) e^{x} + C_2 e^{-2x}$$

Final Solution:

$$y(x) = -(C_1/2) e^{x} + C_2 e^{-2x}$$
$$z(x) = C_1 e^{x} + C_2 e^{-2x}$$

where C_1 and C_2 are arbitrary constants.

Advantages and Limitations of the Elimination Method

Advantages:

- Simplifies complex systems by reducing them to single higher-order equations
- Provides a clear pathway to the general solution
- Suitable for systems with constant coefficients
- Enhances understanding of relationships between variables

Limitations:

- May become cumbersome with non-linear systems
- Not always straightforward if coefficients are variable or complex
- Sometimes leads to higher-order equations that are difficult to solve

Additional Tips for Successfully Applying the Elimination Method

- Always carefully check the algebraic manipulations to avoid errors.
- When differentiating, keep track of derivatives and their associated terms.

- Use substitution carefully to avoid losing track of variables.
- When solving the resulting higher-order differential equations, consider all solution methods available.
- Remember to include the arbitrary constants in the general solution.

Conclusion

The elimination method, when combined with differentiation, is a powerful tool for solving systems of linear differential equations. It provides a systematic approach to reduce complex systems into manageable single-variable differential equations, which can then be solved using standard techniques. By mastering this method, students and practitioners can efficiently find the general solutions of linear systems, facilitating deeper analysis and understanding of various physical, engineering, and mathematical problems.

Whether you're dealing with simple systems or more intricate ones with variable coefficients, the elimination method offers clarity and structure. Practice with diverse examples, and over time, you'll develop an intuitive sense of how to manipulate and solve these systems effectively.

Frequently Asked Questions

How is the elimination method used to find a general solution in a system involving differentiation?

The elimination method involves combining the differential equations to eliminate variables, resulting in a single higher-order differential equation that can be solved to find the general solution of the system.

What are the key steps to apply the elimination method to a linear system involving derivatives?

Key steps include differentiating one equation if necessary, multiplying equations to align coefficients, adding or subtracting equations to eliminate a variable, and then solving the resulting differential equation for the remaining variable.

Can the elimination method be used for nonlinear systems involving derivatives?

While primarily suited for linear systems, the elimination method can sometimes be extended to certain nonlinear systems by suitable substitutions or transformations, but it is more complex and often less straightforward than for linear systems.

What challenges might arise when using the elimination

method with differential equations?

Challenges include managing higher-order derivatives, ensuring proper differentiation and combination of equations, and handling initial conditions. Sometimes, the resulting differential equation can be complicated to solve analytically.

How does the elimination method help in finding the general solution for systems involving differentiation compared to substitution?

The elimination method systematically reduces the system to a single differential equation, which can be more straightforward than substitution when dealing with multiple equations, especially when variables are difficult to isolate directly.

Additional Resources

Use The Elimination Method To Find A General Solution For The Given Linear System, Where Differentiation

Understanding how to solve linear systems of differential equations is fundamental in various fields such as engineering, physics, and applied mathematics. Among the array of methods available, the elimination method stands out as a systematic approach to reduce complex systems into manageable forms, ultimately enabling the derivation of their general solutions. When these systems involve differentiation, the process combines algebraic manipulation with calculus principles, demanding a nuanced understanding of both domains. This article aims to explore the use of the elimination method in solving such systems, providing a comprehensive, detailed, and analytical perspective suitable for students, educators, and practitioners alike.

Understanding the Foundation: Linear Systems of Differential Equations

What Are Linear Systems of Differential Equations?

A linear system of differential equations involves multiple functions and their derivatives, arranged in a set of equations where each equation is linear with respect to the unknown functions and their derivatives. The general form for a system involving two functions $y_1(t)$ and $y_2(t)$ is:

$$
\begin{cases}
a_{11}(t) \frac{dy_1}{dt} + a_{12}(t) y_1 + b_1(t) y_2 = c_1(t) \\
a_{21}(t) \frac{dy_2}{dt} + a_{22}(t) y_2 + b_2(t) y_1 = c_2(t)
\end{cases}
$$

where $a_{ij}(t)$, $b_i(t)$, and $c_i(t)$ are known functions of t.

In many real-world applications, especially those involving physical systems, these equations model phenomena such as electrical circuits, mechanical vibrations, or population dynamics.

The Goal: General Solution

The ultimate goal when solving such systems is to find the general solution, which encompasses all possible particular solutions and includes arbitrary constants reflecting initial conditions. The general solution provides a complete description of the system's behavior over time.

The Elimination Method: Conceptual Overview

What Is the Elimination Method?

The elimination method involves manipulating the system of equations to eliminate one of the unknown functions, resulting in a single differential equation in terms of the remaining function. Once this reduced equation is solved, the eliminated variable can be recovered through back-substitution.

This approach is akin to the method used in algebra for solving systems of linear equations, where one variable is eliminated to solve for the other. In the context of differential equations, the process is more involved because derivatives are present, requiring careful differentiation and algebraic manipulation.

Advantages of the Elimination Method

- Systematic Reduction: It simplifies the system into a single differential equation.
- Applicability: Useful for systems where substitution is complicated or cumbersome.
- Insightful: Helps reveal the structure and interdependence of the variables.

Limitations

- Can become algebraically intensive, especially for systems with variable coefficients.
- Not always the most straightforward method for nonlinear systems.
- Requires the system to be linear and well-posed for elimination.

Step-by-Step Procedure for Applying the Elimination

Method

Applying the elimination method to a system involving differentiation involves several carefully orchestrated steps. Each step is aimed at manipulating the equations to isolate one variable's derivatives, enabling the derivation of a higher-order differential equation for a single unknown function.

Step 1: Write the System Clearly
Begin with the system:

$$
\begin{cases}
a_{11}(t) \frac{dy_1}{dt} + a_{12}(t) y_1 + b_1(t) y_2 = c_1(t) \quad (1) \\
a_{21}(t) \frac{dy_2}{dt} + a_{22}(t) y_2 + b_2(t) y_1 = c_2(t) \quad (2)
\end{cases}
$$

Step 2: Express One Variable in Terms of the Other
Choose one of the equations to solve for one variable or its derivative. For example, solve equation (1) for y_2 or $\frac{dy_1}{dt}$. If the coefficients are non-zero, algebraic manipulation can be performed:

$$
a_{11}(t) \frac{dy_1}{dt} = c_1(t) - a_{12}(t) y_1 - b_1(t) y_2
$$

Similarly, rearranged:

$$
b_1(t) y_2 = c_1(t) - a_{12}(t) y_1 - a_{11}(t) \frac{dy_1}{dt}
$$

which gives:

$$
y_2 = \frac{c_1(t) - a_{12}(t) y_1 - a_{11}(t) \frac{dy_1}{dt}}{b_1(t)}
$$

Alternatively, if y_2 is easier to eliminate, perform similar steps on the other equation.

Step 3: Differentiate as Needed
Differentiate the expression for y_2 with respect to t, as the derivatives will appear in the other equation. This step requires applying the chain rule:

$$
\frac{dy_2}{dt} = \text{Derivative of RHS with respect to } t
$$

This process introduces derivatives of y_1 and known functions.

Step 4: Substitute Back
Replace y_2 and $\frac{dy_2}{dt}$ into the second equation (or the chosen equation), leading to a differential equation solely in terms of y_1 and its derivatives. This step effectively eliminates y_2.

Step 5: Formulate the Higher-Order Differential Equation
As a result of substitution, you will obtain an n-th order ordinary differential equation (commonly second order for two variables), which can be written explicitly:

$$
A(t) \frac{d^2 y_1}{dt^2} + B(t) \frac{dy_1}{dt} + C(t) y_1 = D(t)
$$

where $A(t), B(t), C(t), D(t)$ are functions derived during the algebraic manipulation.

Step 6: Solve the Reduced Differential Equation
Use standard techniques for solving linear differential equations with variable coefficients, such as:

- Variation of parameters
- Reduction of order
- Series solutions
- Recognizing special forms

The solution will involve arbitrary constants corresponding to the order of the differential equation.

Step 7: Back-Substitute for Remaining Variables
Once $y_1(t)$ is determined, substitute back into the earlier expressions to find $y_2(t)$. These functions, combined with the arbitrary constants, form the general solution to the original system.

Application of the Elimination Method: A Worked Example

Let's consider a concrete example to contextualize the steps.

System:

$$
\begin{cases}
\frac{dy_1}{dt} + y_2 = t \quad (1) \\
2 \frac{dy_2}{dt} + y_1 = t^2 \quad (2)
\end{cases}
$$

Step 1: Solve (1) for y_2:

$$

$$
y_2 = t - \frac{dy_1}{dt}
$$

Step 2: Differentiate (2) to express $\frac{dy_2}{dt}$:

$$
2 \frac{dy_2}{dt} + y_1 = t^2
$$

$$
\Rightarrow \frac{dy_2}{dt} = \frac{t^2 - y_1}{2}
$$

But from earlier, $y_2 = t - \frac{dy_1}{dt}$. Differentiating this:

$$
\frac{dy_2}{dt} = 1 - \frac{d^2 y_1}{dt^2}
$$

Step 3: Equate the two expressions for $\frac{dy_2}{dt}$:

$$
1 - \frac{d^2 y_1}{dt^2} = \frac{t^2 - y_1}{2}
$$

Rearranged:

$$
2 - 2 \frac{d^2 y_1}{dt^2} = t^2 - y_1
$$

$$
\Rightarrow 2 - t^2 = - y_1 + 2 \frac{d^2 y_1}{dt^2}
$$

$$
\Rightarrow 2 \frac{d^2 y_1}{dt^2} - y_1 = 2 - t^2
$$

This is a second-order linear differential equation with constant coefficients:

$$
\boxed{
2 y_1'' - y_1 = 2 - t^2
}
$$

Step 4: Solve the homogeneous equation:

$$
2 y_1'' - y_1 = 0
$$

\]

Characteristic equation:

\[
2 r^2 -

Use The Elimination Method To Find A General Solution For The Given Linear System Where Differentiation

Use The Elimination Method To Find A General Solution For The Given Linear System Where Differentiation

Related Articles

- What Volume Of 0.140 MNaOHMNaOH Is Required To Reach The Equivalence Point In The Titration Of 40.0 MLmL
- The Nurse Monitoring A Client Receiving Peritoneal Dialysis Notes That The Client's Outflow Is Less Than
- Tin Atoms Are Introduced Into An FCC Copper ,producing An Alloy With A Lattice Parameter Of 4.758910-8cm

Back to Home