Use The Intermediate Value Theorem To Show That There Is A Solution Of The Equation, X 4x = 1 In (-1,0).

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The Intermediate Value Theorem (IVT) is a fundamental concept in calculus that provides a powerful tool for establishing the existence of solutions to equations within a specific interval. In this article, we will explore how to apply the IVT to demonstrate that the equation $x^{4x} = 1$ has at least one solution in the interval $(-1, 0)$. This example not only illustrates the practical utility of the theorem but also deepens understanding of continuous functions and their properties.

Understanding the Problem and the Equation

Before diving into the application of the IVT, it is essential to understand the key components of the problem.

Analyzing the Equation $x^{4x} = 1$

The equation involves an exponential expression with a variable in both the base and the exponent:

$$
x^{4x} = 1
$$

Our goal is to determine whether there exists some $x \in (-1, 0)$ such that this equation holds true.

Key observations:

- When is $x^{4x} = 1$?
- How does the function behave over the interval $(-1, 0)$?
- What are the properties of the function that allow the application of the IVT?

Reformulating the Problem

To utilize the IVT, which applies to continuous functions, we define a function:

$$
f(x) = x^{4x} - 1

\]

Our task reduces to showing that $f(x) = 0$ for some $x \in (-1, 0)$.

If we can find two points a and b in $(-1, 0)$ such that $f(a)$ and $f(b)$ have opposite signs, and if f is continuous on $[a, b]$, then by the IVT, there exists some $c \in (a, b)$ with $f(c) = 0$.

Understanding Continuity and Behavior of $f(x)$

Is $f(x)$ Continuous on $(-1, 0)$?

The function $f(x) = x^{4x} - 1$ involves a real power of x. For x^{4x} to be well-defined and continuous, we need to consider the domain carefully.

- When $x \in (-1, 0)$, x is negative.
- Raising a negative number to a real power can be problematic unless the exponent is rational with an odd denominator, but since $4x$ is real, the expression x^{4x} is generally undefined for negative x unless we interpret the power in terms of complex logarithms.

However, for the purpose of applying the IVT, we often consider the function:

$$
f(x) = e^{4x \ln |x|} - 1
$$

which extends the definition to real numbers $x \neq 0$, but this involves complex considerations of the logarithm. Alternatively, because the base x is negative over $(-1, 0)$, the function x^{4x} is not defined as a real function for all x in $(-1, 0)$.

But, in many calculus courses and contexts, the problem assumes the principal real-valued branch of x^{4x} defined for negative x with rational exponents, or interprets the function in a way that allows it to be continuous on the interval of interest.

Therefore, to proceed, we interpret $f(x) = x^{4x}$ as the function $e^{4x \ln|x|}$, which is continuous on $(-1, 0)$. Since $\ln|x|$ is continuous on $(-1, 0)$, and the exponential function is continuous everywhere, the composite function $f(x) = e^{4x \ln|x|}$ is continuous on $(-1, 0)$.

Summary:

- $f(x) = e^{4x \ln|x|} - 1$
- Continuous on $(-1, 0)$

With this understanding, we now proceed to identify specific points where $f(x)$ takes values of opposite signs.

Applying the Intermediate Value Theorem

The core strategy involves:

1. Selecting two points a and b within $(-1, 0)$.
2. Computing $f(a)$ and $f(b)$.
3. Showing that $f(a)$ and $f(b)$ have opposite signs.
4. Concluding that there exists $c \in (a, b)$ such that $f(c) = 0$.

Choosing the Points a and b

Let's examine specific points in $(-1, 0)$:

- At $x = -\frac{1}{2}$:

$$
f(-\tfrac{1}{2}) = e^{4 \times (-\tfrac{1}{2}) \times \ln(1/2)} - 1
$$

Note that $\ln(1/2) = -\ln 2$, so:

$$
f(-\tfrac{1}{2}) = e^{(-2) \times (-\ln 2)} - 1 = e^{2 \ln 2} - 1 = e^{\ln 2^2} - 1 = 2^2 - 1 = 4 - 1 = 3
$$

So, $f(-\frac{1}{2}) = 3$, which is positive.

- At $x = -0.9$:

$$
f(-0.9) = e^{4 \times (-0.9) \times \ln 0.9} - 1
$$

Since $\ln 0.9 \approx -0.10536$:

$$
4 \times (-0.9) \times (-0.10536) = 4 \times -0.9 \times -0.10536 \approx 4 \times 0.094824 = 0.3793
$$

Therefore:

$$
f(-0.9) = e^{0.3793} - 1 \approx 1.460 - 1 = 0.460
$$

Positive again.

- At $x = -0.99$:

$$
\ln 0.99 \approx -0.01005
$$

Then:

$$
4 \times (-0.99) \times (-0.01005) \approx 4 \times 0.99 \times 0.01005 \approx 4 \times 0.00995 = 0.0398
$$

So:

$$
f(-0.99) = e^{0.0398} - 1 \approx 1.0406 - 1 = 0.0406
$$

Still positive.

- At $x = -1$:

$$
\ln 1 = 0
$$

So:

$$
f(-1) = e^{0} - 1 = 1 - 1 = 0
$$

At $x = -1$, $f(-1) = 0$.

Note: Since $x = -1$ yields $f(-1)=0$, it is a root. But the problem asks for a solution in $(-1, 0)$, excluding -1 itself.

Now, check at a point closer to zero, say $x = -0.1$:

$$
\ln 0.1 = -2.3026
$$
$$
4 \times (-0.1) \times (-2.3026) = 4 \times -0.1 \times -2.3026 = 4 \times 0.23026 = 0.921
$$
$$
f(-0.1) = e^{0.921} - 1 \approx 2.512 - 1 = 1.512
$$
Positive again.

Now, at $x = -0.0001$:

$$
\ln 0.0001 = -9.2103
$$

$$
4 \times (-0.0001) \times (-9.2103) = 4 \times 0.0001 \times 9.2103 = 4 \times 0.00092103 = 0.003684
$$

$$
f(-0.0001) = e^{0.003684} - 1 \approx 1
$$

Frequently Asked Questions

What is the Intermediate Value Theorem (IVT)?

The Intermediate Value Theorem states that if a continuous function f(x) is defined on a closed interval [a, b], and if d is any number between f(a) and f(b), then there exists some c in (a, b) such that f(c) = d.

How do we apply the IVT to the equation x + 4x = 1 on the interval (-1, 0)?

We define a continuous function f(x) = x + 4x - 1, then check the values of f at the endpoints -1 and 0. If f(-1) and f(0) have opposite signs, IVT guarantees a root in (-1, 0).

What is the function we are analyzing for the equation x + 4x = 1?

The function is f(x) = x + 4x - 1, which simplifies to f(x) = 5x - 1.

Is the function f(x) = 5x - 1 continuous?

Yes, f(x) = 5x - 1 is a linear function, which is continuous everywhere, including on the interval (-1, 0).

What are the values of f(x) at the endpoints -1 and 0?

f(-1) = 5(-1) - 1 = -5 - 1 = -6, and f(0) = 5(0) - 1 = 0 - 1 = -1.

Do the values of f(-1) and f(0) have opposite signs?

No, both are negative (-6 and -1), so the IVT does not directly guarantee a root on this interval based on these endpoint values.

How can we adjust the interval to find a solution using IVT?

We can check the value of f at other points, such as at x = 0.2, where f(0.2) = 5(0.2) - 1 = 1 - 1 = 0, indicating a root at x=0.2.

Is the solution x=0.2 in the interval (-1, 0)?

No, 0.2 is outside the interval (-1, 0), so we need to find a root within the interval, perhaps by checking other points or refining the interval.

What is the correct approach to show there is a solution in (-1, 0) for x + 4x = 1?

Since f(x) = 5x - 1 is continuous, evaluate at x = -1: f(-1) = -6 (negative), and at x approaching 0 from the left, f(0) = -1 (still negative). Since both are negative, the IVT does not guarantee a root in (-1,0). However, if we consider the original equation: x + 4x = 1, evaluated at x = 0: f(0) = -1, and at x = -0.2: f(-0.2) = 5(-0.2) - 1 = -1 - 1 = -2. To find a solution in (-1, 0), we need to find where the function crosses zero, which occurs at x = 0.2, outside the interval. Therefore, re-expressing the problem or rechecking interval bounds is necessary to locate the solution within (-1, 0).

Additional Resources

Use The Intermediate Value Theorem To Show That There Is A Solution Of The Equation, X 4x = 1 In (-1,0)

The Intermediate Value Theorem (IVT) is a fundamental principle in calculus that provides a powerful tool for establishing the existence of solutions to equations within specific intervals. In this article, we will explore how to utilize the IVT to demonstrate that the equation $x \cdot 4^x = 1$ has at least one solution in the interval $(-1, 0)$. This approach combines a thorough understanding of the theorem with meticulous function analysis, offering a clear pathway to solving similar problems.

Understanding the Intermediate Value Theorem

Definition of the IVT

The Intermediate Value Theorem states that if a function f is continuous on a closed interval $[a, b]$, and if N is any number between $f(a)$ and $f(b)$, then there exists at least one point $c \in [a, b]$ such that $f(c) = N$.

Mathematically, if:
- f is continuous on $[a, b]$,
- $f(a) < N < f(b)$ (or $f(b) < N < f(a)$),

then:
- $\exists c \in (a, b)$ such that $f(c) = N$.

This theorem is particularly useful for proving the existence of solutions without explicitly finding the solution.

Key Features of the IVT

- Continuity is essential: The function must be continuous on the interval.
- Interval endpoints matter: The theorem applies directly to the entire interval $[a, b]$.
- Existence, not uniqueness: IVT guarantees at least one solution, but not how many solutions exist.
- Application in root-finding: Commonly used to show that equations have roots within specific intervals.

Analyzing the Equation $x \cdot 4^x = 1$

Defining the Function

To apply the IVT, we first define a function based on the equation:

$$
f(x) = x \cdot 4^x
$$

Our goal is to show that $f(x) = 1$ has at least one solution in $(-1, 0)$.

Properties of $f(x)$

- Continuity: Since x is a polynomial (linear) and 4^x is an exponential function, both continuous everywhere, their product $f(x)$ is continuous for all real x.
- Differentiability: $f(x)$ is differentiable, but for the IVT, continuity suffices.

Applying the IVT to $f(x)$

Step 1: Evaluate $f(x)$ at the endpoints of the interval $(-1, 0)$

Calculate $f(-1)$:

$$
f(-1) = (-1) \cdot 4^{-1} = -1 \cdot \frac{1}{4} = -\frac{1}{4} = -0.25

$$
$$

Calculate $f(0)$:

$$
f(0) = 0 \cdot 4^0 = 0 \cdot 1 = 0
$$

Step 2: Identify the target value $N = 1$

We need to check if $f(x)$ attains the value 1 in $(-1, 0)$. Since $f(x)$ is continuous on $[-1, 0]$, and $f(-1) = -0.25$ and $f(0) = 0$, observe that:

$$
-0.25 < 1 \quad \text{and} \quad 0 < 1
$$

The target value 1 is greater than both $f(-1)$ and $f(0)$.

Step 3: Check the behavior of $f(x)$ within $(-1, 0)$

Since $f(x)$ is continuous, and the values at the endpoints are -0.25 and 0, and considering that $f(x)$ is increasing in $(-1, 0)$ (we'll verify this), we expect that $f(x)$ may cross the value 1 somewhere within $(-1, 0)$.

However, note that at $x = -1$, $f(x) = -0.25$, which is less than 1. At $x=0$, $f(x) = 0$, still less than 1. But we need to see if $f(x)$ ever reaches 1 in $(-1, 0)$. To confirm, examine the function's behavior further.

Finding the Existence of a Solution

Step 4: Analyzing the function's trend

Since $f(x) = x \cdot 4^x$, let's analyze whether $f(x)$ can attain the value 1 between -1 and 0.

Observe that:
- $f(-1) = -0.25$,
- $f(0) = 0$.

The function values are both less than 1, so this suggests that $f(x)$ does not reach 1 at the endpoints. But what about in between?

To proceed, evaluate $f(x)$ at some point in $(-1, 0)$, say $x = -0.5$:

$$
f(-0.5) = -0.5 \cdot 4^{-0.5} = -0.5 \cdot \frac{1}{4^{0.5}} = -0.5 \cdot \frac{1}{2} = -0.5 \cdot 0.5 = -0.25
$$

Again, $f(-0.5) = -0.25 < 1$.

Similarly, at $x = -0.1$:
$$
f(-0.1) = -0.1 \cdot 4^{-0.1} \approx -0.1 \cdot 4^{-0.1}
$$
Calculate $4^{-0.1}$:
$$
4^{-0.1} = e^{-0.1 \ln 4} \approx e^{-0.1 \times 1.386} \approx e^{-0.1386} \approx 0.870
$$
So:
$$
f(-0.1) \approx -0.1 \times 0.870 = -0.087
$$

Again, less than 1. This suggests that $f(x)$ remains below 1 throughout $(-1, 0)$. But wait, the previous calculations show $f(x)$ remains negative or close to zero but never reaches 1.

But here's the key insight: Since $f(x)$ is continuous on $[-1, 0]$, and its maximum value in this interval is less than 1, perhaps there's no solution? Or is there a different approach?

Refining the Approach: Consider a Different Interval

Given the previous reasoning, perhaps the solution is not in $(-1, 0)$. Let's check the value of $f(x)$ at points slightly beyond -1, say at $x = -0.75$:

$$
f(-0.75) = -0.75 \cdot 4^{-0.75}
$$
Calculate $4^{-0.75}$:

$$
4^{-0.75} = e^{-0.75 \ln 4} = e^{-0.75 \times 1.386} \approx e^{-1.0395} \approx 0.353
$$

Thus:

$$
f(-0.75) \approx -0.75 \times 0.353 \approx -0.265
$$

Again, less than 1. So, perhaps the function $f(x)$ never attains 1 in $(-1, 0)$.

But wait — the initial interval considered was $(-1, 0)$. The previous calculations do not show $f(x)$ crossing 1 there. To show the existence of a solution, perhaps we need to look outside $(-1, 0)$.

Alternative Interval: \(\left(0, 1\right)\)

Let's evaluate \(f(1)\):
\[
f(1) = 1 \cdot 4^{1} = 4
\]
At \(x=0\):
\[
f(0) = 0
\]
Since \(f(0)=0\) and \(f(1)=4\), and \(f\) is continuous, by the IVT, there exists some \(c \in (0, 1)\) such that \(f(c) = 1\).

Check the intermediate values:
- At \(x=0.25\):
\[
f(0.25) = 0.

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