Use The Laplace Transform To Solve The Given Initial-value Problem. Y' + 6y = F(t), Y(0) = 0, Where F(t)

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Solving differential equations is a fundamental aspect of engineering, physics, and applied mathematics. Among various methods, the Laplace transform stands out for its ability to convert complex differential equations into simpler algebraic equations, making the process of finding solutions more straightforward. In this article, we will explore how to apply the Laplace transform to solve the initial-value problem (IVP):

$$ Y' + 6y = F(t), \quad Y(0) = 0 $$

where $F(t)$ is a given function. Our goal is to understand each step involved in transforming the differential equation, solving for $Y(s)$, and finally obtaining $y(t)$ through the inverse Laplace transform.

Understanding the Initial-Value Problem

Before delving into the solution process, it is essential to comprehend the components of the problem:

- Differential Equation: $Y' + 6y = F(t)$ — a first-order linear differential equation.
- Initial Condition: $Y(0) = 0$ — specifies the value of the solution at $t=0$.
- Function $F(t)$: The forcing function or input, which could be any function defined over $t \geq 0$.

This IVP models a variety of physical systems, such as RC circuits, population dynamics, or mechanical systems with damping, where the response $y(t)$ depends on an external input $F(t)$.

Fundamentals of the Laplace Transform

The Laplace transform converts a time-domain function $f(t)$ into an algebraic function $F(s)$ in the complex frequency domain:

$$ \mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) dt $$

Key properties relevant to solving differential equations include:

- Linearity: $\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$
- Transforms of derivatives:

$$
\mathcal{L}\{f'(t)\} = sF(s) - f(0)
$$

- Inverse Laplace transform: retrieves the original function from $F(s)$.

Applying the Laplace transform to differential equations can convert derivatives into algebraic terms, simplifying solution procedures.

Step-by-Step Solution of the IVP

Let's now go through the process of solving the IVP:

$$ Y' + 6y = F(t), \quad y(0) = 0 $$

using the Laplace transform.

Step 1: Take the Laplace Transform of Both Sides

Applying the Laplace transform to the entire differential equation:

$$
\mathcal{L}\{Y'\} + 6 \mathcal{L}\{y\} = \mathcal{L}\{F(t)\}
$$

Using the property of derivatives:

$$
sY(s) - y(0) + 6Y(s) = F(s)
$$

Given $y(0) = 0$:

$$
sY(s) + 6Y(s) = F(s)
$$

This simplifies to:

$$
(s + 6) Y(s) = F(s)
$$

Step 2: Solve for $Y(s)$

Rearranging:

$$
Y(s) = \frac{F(s)}{s + 6}
$$

This algebraic expression encapsulates the transformed solution.

Step 3: Find the Inverse Laplace Transform to Obtain $y(t)$

The next step involves applying the inverse Laplace transform to $Y(s)$:

$$
y(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s + 6}\right\}
$$

The specific form of $y(t)$ depends on $F(s)$, the Laplace transform of $F(t)$. Let's explore some common cases.

Case Studies: Solving for Specific $F(t)$

To illustrate the method, we analyze particular forms of $F(t)$, derive their Laplace transforms, and then find the corresponding solutions.

Case 1: $F(t) = \delta(t)$ (Impulse Input)

- Laplace transform:

$$
F(s) = \mathcal{L}\{\delta(t)\} = 1
$$

- Solution in s-domain:

$$
Y(s) = \frac{1}{s + 6}
$$

- Inverse Laplace transform:

$$
y(t) = e^{-6t}
$$

This describes the system's response to an instantaneous impulse at $t=0$.

Case 2: $F(t) = 1$ (Constant Input)

- Laplace transform:

$$
F(s) = \frac{1}{s}
$$

- Solution in s-domain:

$$
Y(s) = \frac{1/s}{s + 6} = \frac{1}{s(s + 6)}
$$

- Partial fraction decomposition:

$$
\frac{1}{s(s + 6)} = \frac{A}{s} + \frac{B}{s + 6}
$$

Solving for A and B:

$$
1 = A(s + 6) + B s
$$

Set $s=0$:

$$
1 = A(6) \Rightarrow A = \frac{1}{6}
$$

Set $s=-6$:

$$
1 = B(-6) \Rightarrow B = -\frac{1}{6}
$$

- Inverse Laplace transform:

$$
y(t) = \frac{1}{6} \cdot 1 + \left(-\frac{1}{6}\right) e^{-6t} = \frac{1}{6} - \frac{1}{6} e^{-6t}
$$

This solution models a steady-state response with transient decay.

General Solution and Interpretation

The general solution for $y(t)$ in the case of arbitrary $F(t)$ is:

$$
y(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s + 6}\right\}
$$

This convolution integral approach can be expressed as:

$$
y(t) = \int_0^{t} f(\tau) \cdot e^{-6(t - \tau)} d\tau
$$

where $f(t) = \mathcal{L}^{-1}\{F(s)\}$.

This integral representation highlights the system's response as a convolution of the input $F(t)$ with the exponential decay e^{-6t}, reflecting the system's damping characteristics.

Practical Applications of the Laplace Transform Method

Applying the Laplace transform to solve initial-value problems is widely used across various fields:

- Electrical Engineering: Analyzing circuits with resistors, capacitors, and inductors.
- Mechanical Systems: Damped oscillators and mass-spring-damper systems.
- Control Systems: Designing and analyzing system responses to inputs.
- Biology: Modeling population dynamics and pharmacokinetics.

The method is particularly advantageous when dealing with piecewise or impulsive inputs, as the Laplace transform simplifies handling initial conditions and discontinuities.

Conclusion

Using the Laplace transform to solve the initial-value problem $Y' + 6y = F(t), \quad y(0) = 0$ provides a powerful, systematic approach to obtaining analytical solutions. By transforming the differential equation into an algebraic equation, solving for $Y(s)$, and applying the inverse Laplace transform, one can derive explicit expressions for $y(t)$ corresponding to various forcing functions $F(t)$.

This method not only simplifies the solution process but also offers insights into the system's behavior through the properties of the Laplace domain, such as poles and zeros. Whether dealing with impulsive forces, constant inputs, or more complex functions, the Laplace transform remains an essential tool in the applied mathematician's toolkit.

Additional Resources and Practice

To deepen understanding, consider exploring the following topics:

- Inverse Laplace Transform Techniques: Partial fractions, convolution theorem.
- Laplace Transform Tables: Common transforms and their inverses.
- Solving Higher-Order Differential Equations: Extension of the method.
- Numerical Methods: When analytical inverse transforms are challenging.

Practicing with various functions $F(t)$ and initial conditions will enhance proficiency in applying the Laplace transform method to differential equations.

Keywords: Laplace transform, initial-value problem, differential equations, algebraic equations, inverse Laplace transform, system response, convolution integral, damping, forced oscillations.

Frequently Asked Questions

What is the first step in using Laplace transforms to solve the initial-value problem Y' + 6Y = F(t), with Y(0) = 0?

The first step is to take the Laplace transform of both sides of the differential equation, converting it into an algebraic equation in terms of Y(s).

How do you apply the Laplace transform to the differential equation Y' + 6Y = F(t)?

Applying the Laplace transform yields sY(s) - Y(0) + 6Y(s) = F(s). Since Y(0) = 0, the equation simplifies to (s + 6)Y(s) = F(s).

What is the role of initial conditions in solving differential equations using Laplace transforms?

Initial conditions are used to evaluate the transformed function, often simplifying the algebraic equations and allowing for the determination of the particular solution.

How do you solve for Y(s) after applying the Laplace transform to the differential equation?

You isolate Y(s) by dividing both sides of the algebraic equation: Y(s) = F(s) / (s + 6).

What is the significance of the inverse Laplace transform in solving the initial-value problem?

The inverse Laplace transform is used to convert Y(s) back into the time domain, providing the solution y(t) to the initial-value problem.

How do you find the inverse Laplace transform of Y(s) = F(s) / (s + 6)?

You use known inverse transforms or partial fraction decomposition to evaluate the inverse Laplace transform, resulting in y(t) = L^{-1} {F(s) / (s + 6)}.

What is the effect of the term (s + 6) in the denominator when solving for Y(s)?

The term $(s + 6)$ indicates a first-order linear differential equation, and it affects the form of the inverse transform, often leading to exponential decay or growth depending on $F(s)$.

Can you give an example of F(t) for which this method is applicable?

Yes, for example, if $F(t) = t$, then $F(s) = 1 / s^2$, and the solution process involves substituting this into $Y(s)$ and performing inverse transforms accordingly.

What are common challenges when using Laplace transforms to solve differential equations like this?

Common challenges include computing the inverse Laplace transform of complex functions, handling partial fractions, and ensuring the initial conditions are correctly applied.

How does the initial condition Y(0) = 0 simplify the solution process?

It eliminates additional terms in the Laplace transform equations, making the algebraic manipulation straightforward and focusing on the relation between $F(s)$ and $Y(s)$.

Additional Resources

Use The Laplace Transform To Solve The Given Initial-value Problem: $Y' + 6y = F(t)$, $Y(0) = 0$

The application of the Laplace transform to solve differential equations is a fundamental technique in engineering and mathematics, especially when dealing with initial-value problems. In this comprehensive review, we will explore the method step-by-step, analyze its advantages and disadvantages, and discuss its practical applications. The specific problem at hand, the first-order linear differential equation $Y' + 6y = F(t)$ with the initial condition $Y(0) = 0$, serves as an excellent example to illustrate the power and elegance of the Laplace transform approach.

Understanding the Laplace Transform

The Laplace transform is an integral transform that converts a time-domain function $f(t)$ into a complex frequency-domain function $F(s)$. It is defined as:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st}f(t) dt
$$

\]

where s is a complex number, $s = \sigma + i \omega$.

Features of the Laplace Transform:
- Converts differential equations into algebraic equations, simplifying the process of solving initial-value problems.
- Handles discontinuous functions, impulses, and step functions effectively.
- Provides a systematic way to incorporate initial conditions directly into the transformed equations.

Applying the Laplace Transform to the Differential Equation

Given the initial-value problem:

$$
Y' + 6y = F(t), \quad Y(0) = 0
$$

we interpret $Y(t)$ as the unknown function to be solved for, and $F(t)$ as a known forcing function.
To solve this using the Laplace transform, follow these steps:

Step 1: Take the Laplace Transform of Both Sides

Applying the Laplace transform to the entire differential equation:

$$
\mathcal{L}\{Y'\} + 6\mathcal{L}\{Y\} = \mathcal{L}\{F(t)\}
$$

Using the linearity property:

$$
sY(s) - Y(0) + 6Y(s) = F(s)
$$

Given $Y(0) = 0$, the equation simplifies to:

$$
sY(s) + 6Y(s) = F(s)
$$

Step 2: Solve for $Y(s)$

Factor out $Y(s)$:

$$
Y(s)(s + 6) = F(s)

\]

Therefore,

$$
Y(s) = \frac{F(s)}{s + 6}
$$

This algebraic expression links the transformed solution $Y(s)$ to the known Laplace transform of $F(t)$.

Step 3: Find the Inverse Laplace Transform

The final step involves applying the inverse Laplace transform to $Y(s)$:

$$
y(t) = \mathcal{L}^{-1}\left\{\frac{F(s)}{s + 6}\right\}
$$

This step heavily depends on the form of $F(s)$. Once $F(s)$ is specified, the inverse transform can be computed either through tables, partial fractions, or other methods.

Example: Solving with a Specific Forcing Function

To illustrate, suppose $F(t) = \delta(t - a)$, an impulse at $t = a$. Its Laplace transform is:

$$
F(s) = e^{-as}
$$

Then, the solution in the s-domain is:

$$
Y(s) = \frac{e^{-as}}{s + 6}
$$

Applying the inverse Laplace transform:

$$
y(t) = \mathcal{L}^{-1}\left\{\frac{e^{-as}}{s + 6}\right\}
$$

Using the shifting property:

$$
\mathcal{L}^{-1}\left\{\frac{e^{-as}}{s + 6}\right\} = u(t - a) e^{-6(t - a)}
$$

where $u(t - a)$ is the Heaviside step function. The solution for this particular forcing function becomes:

$$
y(t) = u(t - a) e^{-6(t - a)}
$$

This example demonstrates how the Laplace transform simplifies the process of solving differential equations with discontinuous or impulsive inputs.

Advantages of Using the Laplace Transform

The Laplace transform method offers several significant benefits:

- Simplification of Differential Equations: Converts differential equations into algebraic equations, which are easier to manipulate and solve.
- Incorporation of Initial Conditions: Initial conditions are directly embedded in the transformed domain, eliminating the need for additional steps.
- Handling of Discontinuities and Impulses: Capable of managing functions like step functions, impulses, and other discontinuous inputs.
- Applicable to a Wide Range of Problems: Suitable for both ordinary and partial differential equations, especially in engineering contexts.
- Systematic Approach: Provides a structured process that can be applied uniformly to various types of problems.

Limitations and Challenges

Despite its strengths, the Laplace transform approach has some limitations:

- Requires Knowledge of Laplace Transforms: Necessitates familiarity with a broad table of transforms and properties.
- Complex Inverse Transforms: In some cases, the inverse transform may involve complex partial fraction decompositions or special functions.
- Not Always the Most Efficient Method: For certain problems, alternative methods like variation of parameters or numerical techniques may be more straightforward.
- Limited to Linear Systems: Primarily applicable to linear differential equations; nonlinear problems are more challenging to handle directly.

Features and Practical Applications

The use of Laplace transforms extends beyond pure mathematics into practical engineering and physics applications:

- Electrical Engineering: Analysis of circuits involving capacitors, inductors, and resistors.

- Mechanical Engineering: Vibration analysis and control systems.
- Control Systems: Designing and analyzing system stability and response.
- Signal Processing: Filtering and system response analysis.
- Physiology and Biological Systems: Modeling biological processes with differential equations.

Conclusion

Using the Laplace transform to solve the initial-value problem $Y' + 6y = F(t)$, with $Y(0) = 0$, exemplifies the transform's power in simplifying differential equations. By converting the problem into the s-domain, solving algebraically, and then transforming back, one can efficiently find solutions even for complex forcing functions. While mastering the method requires familiarity with Laplace transforms and their properties, its advantages—especially in handling initial conditions and discontinuities—make it an invaluable tool in both theoretical and applied contexts. Its versatility in engineering, physics, and mathematics underscores its importance as a fundamental technique for solving linear differential equations with initial conditions.

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