A Function F : Z Z Z Is Defined As F (m,n) = 3n 4m. Verify Whether This Function Is Injective And Whether

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Understanding the properties of functions is a fundamental aspect of mathematical analysis, especially in the field of algebra and number theory. When given a specific function such as $F: Z \times Z \to Z$ defined by $F(m, n) = 3n + 4m$, it becomes crucial to analyze its characteristics—particularly whether it is injective (one-to-one) and surjective (onto). This article aims to provide a comprehensive examination of this function, exploring its injectivity, potential surjectivity, and the mathematical implications of its structure.

Introduction to Functions in Mathematics

Before diving into the analysis of the specific function, it is essential to understand what functions are in the context of mathematics.

Definition of a Function

A function is a relation between a set of inputs (called the domain) and a set of possible outputs (called the codomain), such that each input is related to exactly one output. For the function $F(m, n) = 3n + 4m$, the domain is $Z \times Z$ (the set of all ordered pairs of integers), and the codomain is Z.

Importance of Injectivity and Surjectivity

- Injectivity (One-to-One): A function is injective if different inputs produce different outputs. Formally, F is injective if whenever $F(m_1, n_1) = F(m_2, n_2)$, then $(m_1, n_1) = (m_2, n_2)$.
- Surjectivity (Onto): A function is surjective if every element in the codomain has a pre-image in the domain. That is, for every z in Z, there exists (m, n) in $Z \times Z$ such that $F(m, n) = z$.

Analyzing the Function F(m, n) = 3n + 4m

This section explores the structure of the function, its potential for injectivity, and its surjectivity.

Mathematical Formulation

The function F is defined as:

$$
F(m, n) = 3n + 4m
$$

where m and n are integers.

This is a linear function in two variables, combining m and n linearly with coefficients 4 and 3, respectively.

Question 1: Is F Injective?

To verify whether F is injective, we need to determine if different input pairs (m, n) always produce different outputs, or if there exist distinct pairs that produce the same output.

Step-by-Step Analysis of Injectivity

1. Assume $F(m_1, n_1) = F(m_2, n_2)$:

$$
3n_1 + 4m_1 = 3n_2 + 4m_2
$$

2. Rearranged:

$$
3(n_1 - n_2) = 4(m_2 - m_1)
$$

3. Express as:

$$
3(a) = 4(b)
$$

where $a = n_1 - n_2$ and $b = m_2 - m_1$.

4. Analysis:

$$
3a = 4b
$$

For integers a and b, this equation holds if and only if 3a is divisible by 4, which implies that 3a is a multiple of 4.

5. Divisibility considerations:
Since 3 and 4 are coprime (their greatest common divisor is 1), for the

equation $3a = 4b$ to hold with integers a and b, both sides must be divisible by their common factors. Specifically, for the equality to hold, a must be divisible by 4, and b must be divisible by 3.

6. Conclusion:
- There exist non-zero integer pairs (a, b) satisfying $3a = 4b$ when a and b are multiples of 4 and 3, respectively.
- Therefore, the original pairs (m_1, n_1) and (m_2, n_2) can be different yet produce the same output, indicating that the function is not injective.

Example:
Let's find explicit pairs:
- Choose $a = 4$, then $3 \times 4 = 12$, so $4b = 12 \Rightarrow b=3$.
- Then, $a = n_1 - n_2 = 4$, and $b = m_2 - m_1 = 3$.

Suppose:
- $(m_1, n_1) = (0, 0)$
- $(m_2, n_2) = (3, 4)$

Calculate:
$$
F(0, 0) = 3 \times 0 + 4 \times 0 = 0
$$
$$
F(3, 4) = 3 \times 4 + 4 \times 3 = 12 + 12 = 24
$$
They produce different outputs, so we need another example where the outputs are equal.

Let’s pick:
- $(m_1, n_1) = (0, 0)$
- $(m_2, n_2) = (3, 4)$

We see their outputs are different, so to find pairs with the same output, set:

$$
3n_1 + 4m_1 = 3n_2 + 4m_2
$$

Suppose:
- $m_2 = m_1 + 3$
- $n_2 = n_1 + 4$

Calculate:
$$
F(m_1, n_1) = 3 n_1 + 4 m_1
$$
$$
F(m_2, n_2) = 3(n_1 + 4) + 4(m_1 + 3) = 3 n_1 + 12 + 4 m_1 + 12 = 3 n_1 + 4 m_1 + 24
$$

\]

Since the outputs differ by 24, not equal, this indicates that different pairs can produce different outputs. Alternatively, the key is to find pairs where:

$$
3 n_1 + 4 m_1 = 3 n_2 + 4 m_2
$$

and the pairs $(m_1, n_1) \neq (m_2, n_2)$. Because of the linearity, for any fixed output z, there are infinitely many pairs (m, n) satisfying:

$$
3 n + 4 m = z
$$

which leads to the conclusion that the function is not injective.

Question 2: Is F Surjective?

Next, we analyze whether F is surjective, i.e., for any integer z, can we find integers m and n such that:

$$
F(m, n) = 3 n + 4 m = z
$$

Analysis of Surjectivity

1. Rewrite the equation:
$$
3 n + 4 m = z
$$
2. Goal: For any arbitrary z in Z, find integers m, n satisfying this equation.

3. Method:
- For fixed z, treat it as a linear Diophantine equation in m and n.
- The general form of a linear Diophantine equation $ax + by = c$ has solutions if and only if $\gcd(a, b)$ divides c.

4. Calculate $\gcd(3, 4)$:
$$
\gcd(3, 4) = 1
$$
Since 1 divides every integer z, solutions exist for all z.

5. Explicit solution:
- For any z, choose an arbitrary m, then solve for n:

$$
3 n = z - 4 m
$$

- For n to be integer, $z - 4 m$ must be divisible by 3.

6. Existence of solutions:
- For given z, select m such that:

$$
z - 4 m \equiv 0 \pmod{3}
$$

- Since 4 ≡ 1 (mod 3), the congruence becomes:

$$
z - m \equiv 0 \pmod{3}
$$

- Therefore:

$$
m \equiv z \pmod{3}
$$

- For each z, choose $m \equiv z \pmod{3}$, for example:

$$
m = z \quad \text{(if} \quad z \equiv 0 \pmod{3}\text{)}
$$

or
\[
m = z - 1 \

Frequently Asked Questions

What is the definition of the function F : Z × Z → Z given by F(m, n) = 3n + 4m?

The function is defined as F(m, n) = 3n + 4m, where m and n are integers, mapping from pairs of integers to integers.

How do you determine if the function F is injective?

To check injectivity, you need to verify that if F(m1, n1) = F(m2, n2), then (m1, n1) = (m2, n2).

Is the function F(m, n) = 3n + 4m injective? Why or why not?

No, F is not injective because different pairs (m, n) can produce the same value. For example, F(0, 1) = 3 and F(3, 0) = 12, which are different, but more generally, there exist distinct pairs with the same output.

How can we test whether the function F is surjective?

To test surjectivity, verify whether for every integer z, there exist integers m and n such that F(m, n) = z.

Is the function F surjective over Z? Why or why not?

Yes, F is surjective because for any integer z, we can find integers m and n such that z = 3n + 4m. For example, fixing m = 0, n = z/3 (if z divisible by 3) or choosing appropriate m and n to satisfy the equation.

What is the significance of the coefficients 3 and 4 in the function regarding injectivity and surjectivity?

The coefficients 3 and 4 are integers with a greatest common divisor of 1, which influences the function's properties: it makes the function surjective over Z but not injective, since multiple pairs can map to the same value.

Based on the analysis, is the function F injective, surjective, both, or neither?

The function F is surjective but not injective over Z.

Additional Resources

A Function F : Z × Z → Z is Defined As F(m, n) = 3n + 4m. Verify Whether This Function Is Injective And Whether

Understanding the nature of functions between different sets, especially from integers to integers, is fundamental in mathematical analysis and algebra. The function in question, F: Z × Z → Z defined by F(m, n) = 3n + 4m, presents an interesting case for investigation regarding its properties, particularly injectivity. This article aims to thoroughly analyze whether this function is injective, exploring the core concepts, providing detailed proofs, counterexamples, and discussing its implications.

Understanding the Function F : Z × Z → Z

Before delving into the properties of injectivity, it is essential to understand the structure and behavior of the function.

Definition:
$F(m, n) = 3n + 4m$, where m and n are integers.

This is a bilinear form, specifically a linear combination of the variables m and n. For each ordered pair (m, n), the function outputs an integer calculated as thrice the second component plus four times the first component.

Features:

- Linear in both variables: For fixed m, the function is linear in n, and vice versa.
- Surjective or not? The function's range covers all integers because, for any integer k, one can find suitable m and n to produce k.

Verifying Whether F Is Injective

Injectivity (or one-to-one property) is a fundamental aspect of functions. A function F is injective if and only if:

> For all $(m_1, n_1), (m_2, n_2) \in Z \times Z$,
> if $F(m_1, n_1) = F(m_2, n_2)$, then $(m_1, n_1) = (m_2, n_2)$.

In other words, different inputs should produce different outputs; no two distinct input pairs should map to the same value.

Formal Definition and Approach

To verify whether F is injective, we analyze the condition:

$F(m_1, n_1) = F(m_2, n_2) \Rightarrow (m_1, n_1) = (m_2, n_2)$.

Substituting the function:

$3n_1 + 4m_1 = 3n_2 + 4m_2$.

Rearranged, this becomes:

$3(n_1 - n_2) = 4(m_2 - m_1)$.

This equation is central to our analysis. It relates the differences in the inputs to the differences in the outputs.

Analysis of the Equation $3(n_1 - n_2) = 4(m_2 - m_1)$

The key insight is that the differences in the variables must satisfy a linear relation:

$3\Delta n = 4\Delta m$,

where $\Delta n = n_1 - n_2$ and $\Delta m = m_2 - m_1$.

For the function to be injective, the only solution to this equation should be $\Delta n = 0$ and $\Delta m = 0$, meaning the inputs are identical.

Existence of Non-Trivial Solutions

Let's analyze whether non-trivial solutions exist:

- Suppose $\Delta n = 4k$, and $\Delta m = 3k$, for some integer k.

- Plugging into the equation:

$3(4k) = 4(3k)$
$12k = 12k$,

which holds for all integers k.

This means that for any integer k, the pair of differences $(\Delta m, \Delta n) = (3k, 4k)$ satisfies the equation, even when $(\Delta m, \Delta n) \neq (0, 0)$.

Implication:
Two different input pairs, (m_1, n_1) and (m_2, n_2), with:

$m_2 = m_1 + 3k$,
$n_2 = n_1 + 4k$,

will produce the same output:

$F(m_1, n_1) = F(m_2, n_2)$.

Example:
Let’s take specific values:

- $(m_1, n_1) = (0, 0)$, then

$F(0, 0) = 0$.

- For $k = 1$,

$(m_2, n_2) = (0 + 31, 0 + 41) = (3, 4)$,

then

$F(3, 4) = 34 + 43 = 12 + 12 = 24$.

Check the initial pair:

$F(0, 0) = 0$, and the second pair:

$F(3, 4) = 34 + 43 = 12 + 12 = 24$.

Wait, this suggests an inconsistency; the outputs are different, so perhaps a specific example with $k=1$:

- $(m_1, n_1) = (0, 0)$,

- $(m_2, n_2) = (3, 4)$,

then $F(0, 0) = 0$,

$F(3, 4) = 34 + 43 = 12 + 12 = 24$.

They are different, so the previous conclusion needs refining.

Actually, the core relation is:

$3(n_1 - n_2) = 4(m_2 - m_1)$.

Given that, if:

$m_2 - m_1 = 3k$,

$n_2 - n_1 = 4k$,

then the outputs:

$F(m_2, n_2) = 3n_2 + 4m_2$,

$= 3(n_1 + 4k) + 4(m_1 + 3k)$,

$= 3n_1 + 12k + 4m_1 + 12k$,

$= (3n_1 + 4m_1) + 24k$,

which is equal to $F(m_1, n_1) + 24k$.

Thus, the outputs differ by 24k, unless k=0.

Conclusion from this:
The initial assumption that the function is not injective was incorrect because different pairs can produce different outputs unless the difference in inputs produces the same output.

Re-expressing, the correct analysis is:

- For the equation $3(n_1 - n_2) = 4(m_2 - m_1)$, solutions exist when the right side is divisible by 3.

- Since $4(m_2 - m_1)$ must be divisible by 3, and 4 is not divisible by 3, the only way is that $(m_2 - m_1)$ is divisible by 3, and the corresponding difference in n must satisfy $3(n_1 - n_2) = 4(m_2 - m_1)$.

- For arbitrary m_2, m_1, unless $(m_2 - m_1)$ is divisible by 3, the equality cannot hold, implying the function is not necessarily injective across all inputs.

Final Verdict on Injectivity

Since:

- For arbitrary pairs $(m_1, n_1) \neq (m_2, n_2)$, the equation $3(n_1 - n_2) = 4(m_2 - m_1)$ may hold for some specific differences, meaning multiple input pairs can yield the same output.

- For example, choosing $(m_1, n_1) = (0, 0)$, $(m_2, n_2) = (3, 4)$:

F(0, 0) = 0,

F(3, 4) = 34 + 43 = 12 + 12 = 24,

which is different, so these do not map to the same output, but if we consider $(m_2, n_2) = (3, 4)$:

- For the same output, 3n + 4m = k,

- For different pairs, the outputs are different unless the pairs satisfy the relation:

$3(n_1 - n_2) = 4(m_2 - m_1)$.

Thus, the function is not injective because there exist distinct pairs (m, n) and (m + 3k, n + 4k) that could produce the same or different outputs depending on the value of k.

Explicitly, the function is not injective because:

- Multiple pairs can produce the same output.

- For example, (m, n) and (m + 3, n + 4):

$F(m, n) = 3n + 4m$,

$F(m + 3, n + 4) = 3(n + 4) + 4(m + 3) = 3n + 12 + 4m + 12 = 3n + 4m + 24$.

They produce outputs differing by 24, so not the same, unless the difference is zero. But if we consider the inverse, for different pairs, the outputs are generally different unless the pairs are related by the specific linear relation.

Hence, the function F is not injective.

Summary of Key Findings

- The function $F(m, n) = 3n + 4m$ is a bilinear form involving linear combinations of m and n.

- The analysis

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