Apply The Laplace Transform To Solve The Following Initial Value Problem Y" + 4y = 0, Y(0) = 1, And Y'(0)

Apply The Laplace Transform To Solve The Following Initial Value Problem Y" + 4y = 0, Y(0) = 1, And Y'(0)

Solving differential equations is a fundamental aspect of mathematical analysis, especially in engineering and physics. One powerful technique utilized for solving linear differential equations with constant coefficients and initial conditions is the Laplace Transform. This method simplifies complex differential equations into algebraic equations, making them easier to manipulate and solve. In this article, we will explore how to apply the Laplace Transform to solve the initial value problem (IVP):

$$Y'' + 4Y = 0$$
with initial conditions:
$$Y(0) = 1, \quad Y'(0) = \text{(to be determined)}$$

Our goal is to determine the function $Y(t)$ that satisfies this differential equation along with the given initial conditions. We will walk through the process step-by-step, explaining the theory behind the Laplace Transform, its application to the differential equation, and the interpretation of the solution.

Understanding the Differential Equation and Initial Conditions

Before applying the Laplace Transform, it's essential to understand the nature of the differential equation and the initial conditions.

The Differential Equation: $Y'' + 4Y = 0$

- This is a second-order linear homogeneous differential equation with constant coefficients.
- The characteristic equation associated with it is:

$$r^2 + 4 = 0$$

- The roots of the characteristic equation are complex:

$$r = \pm 2i$$

- The general solution to this differential equation, based on the roots, is:

$$ Y(t) = C_1 \cos 2t + C_2 \sin 2t $$

where C_1 and C_2 are constants determined by initial conditions.

Initial Conditions and Unknowns

- We are given $Y(0) = 1$, which allows us to find C_1 once C_2 is known.
- The initial value of $Y'(0)$ is not explicitly given in the problem statement, but typically, in such problems, it's either provided or assumed to be zero unless otherwise specified.

Assuming the initial derivative $Y'(0)$ is known or to be determined, the steps to proceed involve transforming the differential equation into an algebraic form using the Laplace Transform, applying initial conditions, solving for $Y(t)$, and then interpreting the solution.

Introduction to the Laplace Transform

The Laplace Transform is a mathematical operation that transforms a time-domain function $y(t)$ into a complex frequency domain function $Y(s)$. It is particularly useful for solving linear differential equations with initial conditions.

Definition of the Laplace Transform

The Laplace Transform of a function $y(t)$, defined for $t \ge 0$, is:

$$ \mathcal{L}\{y(t)\} = Y(s) = \int_0^{\infty} e^{-st} y(t) \, dt $$

where:

- s is a complex variable $s = \sigma + i\omega$,
- the integral converges for values of s where $y(t)$ is of exponential order.

Key Properties of the Laplace Transform

Some properties that facilitate solving differential equations include:

- Linearity: $\mathcal{L}\{a y_1 + b y_2\} = a Y_1(s) + b Y_2(s)$
- Derivative property:

$$ \mathcal{L}\{ y'(t) \} = s Y(s) - y(0) $$

$$ \mathcal{L}\{ y''(t) \} = s^2 Y(s) - s y(0) - y'(0) $$

- Initial value theorem: $y(0) = \lim_{s \to \infty} s Y(s)$

These properties allow us to convert differential equations into algebraic equations in the s-domain.

Applying the Laplace Transform to the Differential Equation

To solve the initial value problem, we take the Laplace Transform of both sides of the differential equation:

$$Y'' + 4Y = 0$$

with the initial conditions $Y(0) = 1$ and $Y'(0) = y'_0$ (unknown or specified).

Step 1: Take the Laplace Transform of Both Sides

Applying the Laplace Transform:

$$\mathcal{L}\{Y''\} + 4 \mathcal{L}\{Y\} = 0$$

Using the derivative property:

$$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = 0$$

Substitute $y(0) = 1$:

$$s^2 Y(s) - s \times 1 - y'_0 + 4Y(s) = 0$$

Rearranged:

$$(s^2 + 4) Y(s) = s + y'_0$$

Assuming y'_0 is known, this algebraic equation can be solved for $Y(s)$:

$$Y(s) = \frac{s + y'_0}{s^2 + 4}$$

If y'_0 is not given, it can be left as a parameter, or, if the problem assumes $Y'(0) = 0$, then $y'_0 = 0$.

Solving for $Y(s)$ and Inverting the Transform

Once $Y(s)$ is expressed, the next step is to find $y(t)$ by taking the inverse Laplace Transform.

Case 1: When $y'_0 = 0$

In this case:

$$Y(s) = \frac{s}{s^2 + 4}$$

This form corresponds to a standard Laplace Transform pair:

$$\mathcal{L}\{\cos 2t\} = \frac{s}{s^2 + 4}$$

Therefore, the inverse Laplace Transform yields:

$$y(t) = \cos 2t$$

Applying the initial condition:

$$y(0) = \cos 0 = 1$$

which matches the given initial value, confirming the solution.

Case 2: When $y'_0 \neq 0$

Suppose y'_0 is specified (say, $y'_0 = 0$ for simplicity), then:

$$Y(s) = \frac{s}{s^2 + 4}$$

and the solution in the time domain is:

$$y(t) = \cos 2t$$

Alternatively, if y'_0 is specified as some value a, then:

$$Y(s) = \frac{s + a}{s^2 + 4}$$

which corresponds to:

$$y(t) = \cos 2t + a \frac{\sin 2t}{2}$$

based on the inverse Laplace Transform pairs:

- $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2 + 4}$

Interpreting the Final Solution

The general solution to the IVP, considering initial conditions, is:

$$ Y(t) = C_1 \cos 2t + C_2 \sin 2t $$

where:

- $C_1 = Y(0) = 1$,
- C_2 is determined from $Y'(0)$, which can be found by differentiating the solution or directly from the inverse Laplace Transform.

If the initial derivative $Y'(0)$ is known, say $Y'(0) = y'_0$, then:

$$ y'(t) = -2 C_1 \sin 2t + 2 C_2 \cos 2t $$

and at $t=0$:

$$ y'(0) = 2 C_2 $$

Thus, $C_2 = \frac{Y'(0)}{2}$.

Summary of the solution:

- The solution is a combination of sine and cosine functions with frequencies determined by the characteristic roots.
- The initial conditions dictate the specific amplitudes.

Practical Applications of Laplace Transform in Solving Differential Equations

The technique of applying the Laplace Transform extends beyond simple second-order equations. It is widely used in various fields:

1. **Electrical Engineering:** Analyzing circuits with capacitors and inductors by transforming differential equations into algebraic equations.

2. **Mechanical Systems:** Solving differential equations governing vibrations and oscillations.

3. **Control Systems:** Designing controllers and analyzing system stability using Laplace domain methods.

4. **Physics:** Solving heat, wave, and quantum mechanics equations with initial conditions.