Assume That T Is A Linear Transformation. Find The Standard Matrix Of T:R2R2T:R 2R 2 First Rotates Points

Assume That T Is A Linear Transformation. Find The Standard Matrix Of T:R2→R2 First Rotates Points

Understanding linear transformations and their matrices is fundamental in linear algebra, especially when analyzing how geometric operations like rotations, reflections, and scalings affect vectors in the plane. In this article, we will explore how to determine the standard matrix of a specific linear transformation T: $\mathbb{R}^2 \to \mathbb{R}^2$ that involves rotation. We will go through the process step-by-step, providing detailed explanations and illustrative examples to clarify the concepts.

Introduction to Linear Transformations and Matrices

Before diving into the specific problem, let's review some essential background information.

What Is a Linear Transformation?

A linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a function that satisfies two properties for all vectors u and v in $\mathbb{R}^2$ and scalar c:

- Additivity: T(u + v) = T(u) + T(v)
- Homogeneity: T(cu) = cT(u)

These properties ensure that T preserves vector addition and scalar multiplication, making it a linear operator.

Standard Matrix of a Linear Transformation

Any linear transformation T: $\mathbb{R}^2 \to \mathbb{R}^2$ can be represented by a 2×2 matrix A such that:

$$T(\mathbf{x}) = A \mathbf{x}$$

for all vectors $\mathbf{x} \in \mathbb{R}^2$. The matrix A is called the standard matrix of T, and its columns are the images of the basis vectors:

$$A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2)]$$

where:

- $\mathbf{e}_1 = (1, 0)$
- $\mathbf{e}_2 = (0, 1)$

Understanding this allows us to analyze and compute the effects of T on any vector in the plane.

Defining the Transformation T: Rotation in $\mathbb{R}^2$

The problem states that T is a linear transformation that first rotates points.

What Does Rotation in the Plane Mean?

- Rotation is a transformation that turns every point in the plane around the origin by a fixed angle θ.
- The rotation matrix R_θ is a standard matrix that performs this operation.

Rotation Matrix in $\mathbb{R}^2$

The rotation matrix R_θ for an angle θ (counterclockwise) is:

$$
R_\theta =
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

Applying R_θ to a vector rotates it by θ degrees about the origin.

Step-by-Step Process to Find the Standard Matrix of T

Given the description, T first rotates points, and possibly applies other transformations. For this article, we will assume that T is a rotation transformation alone, as per the initial hint.

Step 1: Identify the Rotation Angle

- The problem states that T "first rotates points," but it does not specify the angle.
- For completeness, we will consider a general rotation by an angle θ, which can be any real number.
- If the specific angle is given, replace θ with that value.

Step 2: Write the Rotation Matrix

- Using the general formula, the standard matrix for T is:

$$
A = R_\theta =
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

- This matrix rotates any vector in $\mathbb{R}^2$ by θ degrees CCW.

Step 3: Verify Linearity

- Rotation matrices are linear transformations because they satisfy additivity and homogeneity.
- Confirmed by the properties of matrix multiplication.

Step 4: Apply to Basis Vectors

- To explicitly find the standard matrix, compute T($\mathbf{e}_1$) and T($\mathbf{e}_2$):

1. $T(\mathbf{e}_1) = R_\theta \times \begin{bmatrix}1 \\ 0\end{bmatrix} = \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix}$

2. $T(\mathbf{e}_2) = R_\theta \times \begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}$

- These vectors form the columns of the standard matrix A.

Explicit Form of the Standard Matrix

Putting it all together, the standard matrix A for the rotation transformation T is:

$$
A =
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

This matrix rotates any vector in $\mathbb{R}^2$ by θ degrees CCW.

Special Cases and Examples

Understanding particular cases helps solidify the concept.

Example 1: Rotation by 90 Degrees

- $\theta = 90°$, so:

$$
\cos 90^\circ = 0, \quad \sin 90^\circ = 1
$$

- The matrix becomes:

$$
A =
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
$$

- Applying this matrix to a vector rotates it 90° CCW.

Example 2: Rotation by 180 Degrees

- $\theta = 180°$, so:

$$
\cos 180^\circ = -1, \quad \sin 180^\circ = 0
$$

- The matrix:

$$
A =
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}
$$

- Rotates vectors by 180°, effectively flipping them.

Additional Considerations: Combining Rotation with Other Transformations

While the problem specifies a rotation, sometimes transformations involve multiple steps.

Rotation Followed by Reflection or Scaling

- The combined transformation can be represented by multiplying the matrices.
- For example, if T first rotates then scales, the total matrix is:

$$
A_{total} = S \times R_\theta
$$

where S is the scaling matrix.

Effect on Standard Basis Vectors

- The images of $\mathbf{e}_1$ and $\mathbf{e}_2$ under T fully describe the transformation.
- This approach simplifies understanding the geometric effect.

Summary and Key Takeaways

- The standard matrix for a rotation transformation T in $\mathbb{R}^2$ by angle θ is:

$$
A =
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

- To find the matrix, identify the rotation angle, then compute the images of the basis vectors.
- Rotation matrices are orthogonal and preserve vector lengths and angles.
- These matrices are fundamental in computer graphics, robotics, and physics for modeling rotational movements.

Conclusion

Understanding how to derive the standard matrix of a linear transformation, especially rotations, is crucial in various applications. By recognizing that rotation matrices are well-established and their columns correspond to the images of basis vectors, you can efficiently analyze and implement rotational transformations in a coordinate plane. Whether working with specific angles or general cases, the process remains consistent and straightforward, empowering you to handle more complex geometric transformations with confidence.

Keywords: linear transformation, standard matrix, rotation matrix, $\mathbb{R}^2$, basis vectors, geometric transformations, matrix representation, rotation angle, linear algebra

Frequently Asked Questions

What is the standard matrix of a linear transformation T: $R^2 \rightarrow R^2$ that first rotates points?

The standard matrix of such a transformation is the rotation matrix R(θ) = [[cos θ, -sin θ], [sin θ, cos θ]], where θ is the rotation angle.

How do you determine the matrix of a rotation transformation in R^2?

You determine the matrix by using the rotation formula: R(θ) = [[cos θ, -sin θ], [sin θ, cos θ]], which rotates points by angle θ counterclockwise.

If T first rotates points by 90 degrees counterclockwise, what is its standard matrix?

The matrix for a 90-degree rotation is R(90°) = [[0, -1], [1, 0]].

What properties does the standard matrix of a rotation transformation have?

It is an orthogonal matrix with determinant 1, and it preserves lengths and angles in R^2.

How can you verify that the matrix represents a rotation transformation?

Check that the matrix is orthogonal (its transpose equals its inverse) and has a determinant of 1, indicating a pure rotation.

Can the standard matrix of T be expressed in terms of sine and cosine functions?

Yes, it is expressed as [[cos θ, -sin θ], [sin θ, cos θ]] for some angle θ.

If T is a rotation by 45 degrees, what is its standard matrix?

The matrix is R(45°) = [[cos 45°, -sin 45°], [sin 45°, cos 45°]] = [[√2/2, -√2/2], [√2/2, √2/2]].

What is the effect of applying the standard matrix of a rotation

transformation to a vector in R^2?

It rotates the vector by the specified angle θ counterclockwise around the origin.

How does the standard matrix change if the rotation is clockwise instead of counterclockwise?

The matrix becomes $R(-\theta) = [[\cos\theta, \sin\theta], [-\sin\theta, \cos\theta]]$, which rotates points clockwise by angle θ.

Why is the standard matrix of a rotation transformation orthogonal?

Because rotation preserves lengths and angles, its matrix must be orthogonal, satisfying $R^t R = I$, with determinant 1.

Additional Resources

Linear Transformation: Decoding the Standard Matrix of a Rotation in $\mathbb{R}^2$

Transformations are foundational concepts in linear algebra, acting as the engines behind how vectors are manipulated within various coordinate spaces. Among these, rotation transformations stand out due to their geometric significance and widespread applications—from computer graphics and robotics to physics and engineering. This article provides an expert-level exploration of linear transformations, focusing specifically on rotations in the two-dimensional Euclidean space $\mathbb{R}^2$. We will examine how to determine the standard matrix of such a transformation, assuming the transformation T is a rotation, and how this matrix encapsulates the essence of the rotation operation.

Understanding the Fundamentals: What is a Linear Transformation?

Before delving into rotations, it's essential to revisit what constitutes a linear transformation.

Definition and Properties

A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is a function that satisfies two main properties for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars c:

1. Additivity:
$$
T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})
$$
2. Homogeneity:
$$
T(c \mathbf{u}) = c T(\mathbf{u})
$$

In the specific case of $T: \mathbb{R}^2 \to \mathbb{R}^2$, linear transformations can be represented as matrix multiplications:

$$
T(\mathbf{x}) = A \mathbf{x}
$$

where A is a 2×2 matrix, called the standard matrix of T. This matrix encapsulates all the information about the transformation, including rotation, scaling, reflection, or shear.

Rotations in $\mathbb{R}^2$: Geometric Intuition

A rotation is a transformation that turns every vector about the origin by a fixed angle θ. In the plane, this operation preserves lengths and angles—making it an orthogonal transformation with determinant 1.

Geometric Definition

Given a point $\mathbf{x} = (x, y)$, rotating it by an angle θ counterclockwise about the origin results in a new point $\mathbf{x'}$:

$$
\mathbf{x'} = R_\theta \mathbf{x}
$$

where R_θ is the rotation matrix associated with angle θ.

The Rotation Matrix

The standard matrix for a rotation by an angle θ in $\mathbb{R}^2$ is well-known and given by:

$$
R_\theta =
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

This matrix has several key properties:

- Orthogonality: $R_\theta^T R_\theta = I$
- Determinant: $\det(R_\theta) = 1$
- Inverse: $R_\theta^{-1} = R_{-\theta}$

Determining the Standard Matrix of T: The Rotation Case

Suppose we have a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$, and we're told that T is a rotation. Our goal is to find the standard matrix A such that:

$$
T(\mathbf{x}) = A \mathbf{x}
$$

for all $\mathbf{x} \in \mathbb{R}^2$.

Step 1: Identify the Rotation Angle θ

The key to determining A is understanding the rotation angle θ. Typically, this angle is either given explicitly or can be inferred from the transformation's action on specific vectors.

Example Scenarios:

- If the images of the basis vectors are known, such as:

$$
T\left(\begin{bmatrix}1 \\ 0\end{bmatrix} \right) = \begin{bmatrix} \cos \theta \\ \sin \theta
\end{bmatrix}
$$

and

$$
T\left(\begin{bmatrix}0 \\ 1\end{bmatrix} \right) = \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix}
$$

then the rotation angle θ can be directly deduced from these images.

- If the transformation's effect on points is known, such as how it rotates a specific vector, the angle can be computed via:

$$
\cos \theta = \frac{\mathbf{x} \cdot T(\mathbf{x})}{\|\mathbf{x}\| \|T(\mathbf{x})\|}
$$

or by using inverse trigonometric functions on the components.

Step 2: Construct the Standard Matrix

Once θ is determined, the standard matrix A is precisely the rotation matrix:

$$
A = R_\theta =
\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
$$

This matrix acts on any vector $\mathbf{x}$ to produce its rotated image.

Explicit Example: Computing the Standard Matrix of a Rotation

Let's take a concrete example to illustrate the process.

Given:

Suppose T is a linear transformation on $\mathbb{R}^2$ that rotates any vector by 60 degrees counterclockwise.

Step 1: Identify θ

$$
\theta = 60^\circ
$$

In radians:

$$
\theta = \frac{\pi}{3}
$$

Step 2: Write the Rotation Matrix

$$
A = R_{60^\circ} =
\begin{bmatrix}
\cos \frac{\pi}{3} & -\sin \frac{\pi}{3} \\
\sin \frac{\pi}{3} & \cos \frac{\pi}{3}
\end{bmatrix}
$$

Using known values:

$$
\cos \frac{\pi}{3} = \frac{1}{2}, \quad \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}
$$

Thus,

$$
A =
\begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix}
$$

This matrix fully describes the linear transformation T, rotating every vector in the plane by 60 degrees counterclockwise.

Properties and Significance of the Rotation Matrix

Understanding the properties of the standard matrix of a rotation offers insights beyond mere computation.

Orthogonality and Length Preservation

Because R_θ is an orthogonal matrix:

- It preserves the Euclidean norm:

$$
\| R_\theta \mathbf{x} \| = \| \mathbf{x} \|
$$

- It preserves angles between vectors, making it an isometry.

Determinant

The determinant being 1 indicates an orientation-preserving transformation.

Inverse

The inverse of a rotation matrix is associated with a rotation by $-\theta$:

$$
R_\theta^{-1} = R_{-\theta}
$$

which is obtained by replacing θ with $-\theta$ in the matrix.

Applications and Broader Context

The standard matrix of rotation in $\mathbb{R}^2$ isn't just a theoretical construct; it underpins numerous practical applications:

- Computer Graphics: Rotating images, objects, and camera views.
- Robotics: Planning and controlling the orientation of robotic arms.
- Physics: Analyzing rotational motion and angular momentum.

- Signal Processing: Implementing phase shifts and frequency domain transformations.

Moreover, understanding how to derive these matrices is fundamental for more complex transformations, such as combined rotations and scalings, reflections, or affine transformations.

Extending to Other Transformations

While our focus has been on pure rotations, linear transformations in $\mathbb{R}^2$ can be more complex, involving:

- Reflections: Matrices with determinant -1, such as reflection across a line.
- Shear transformations: Matrices that skew the plane.
- Scaling: Diagonal matrices stretching or compressing axes.

Knowing how to identify the standard matrix for a rotation provides a foundation for understanding and constructing these other transformations.

Conclusion

In summary, when a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a rotation, its

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