Compute The Following Derivatives, Showing All Work As Required. A. Using First Principles, Differentiate

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Introduction to Derivatives and First Principles

Understanding derivatives is fundamental in calculus, providing the tools necessary to analyze how functions change at any given point. Derivatives are essential in various fields such as physics, engineering, economics, and more, where they describe rates of change, slopes of curves, and optimization problems.

One of the most rigorous approaches to derivative calculation is using first principles, also known as the definition of the derivative. This approach involves understanding derivatives as the limit of the average rate of change of a function over an interval as the interval approaches zero.

In this article, we will explore how to compute derivatives from first principles, demonstrating all necessary steps. We'll include detailed explanations, step-by-step calculations, and useful tips to master the process.

Understanding the First Principles Definition of Derivative

The derivative of a function $f(x)$ at a point $x=a$ using first principles is defined as:

$$
f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
$$

This limit, if it exists, gives the instantaneous rate of change of the function at $x=a$. The process involves:

1. Calculating $f(a+h)$.
2. Finding the difference $f(a+h) - f(a)$.
3. Dividing this difference by the small change h.
4. Taking the limit as h approaches zero.

This method provides a fundamental understanding of derivatives, emphasizing their geometric interpretation as the slope of the tangent line to the curve at a point.

Step-by-Step Guide to Computing Derivatives Using First Principles

Below are the general steps to differentiate a function from first principles:

1. Identify the function $f(x)$.
2. Substitute $x=a+h$ into the function to find $f(a+h)$.
3. Compute the difference $f(a+h) - f(a)$.
4. Form the difference quotient $\frac{f(a+h) - f(a)}{h}$.
5. Simplify the expression as much as possible.
6. Evaluate the limit as $h \to 0$.
7. The resulting expression is $f'(a)$.

Example 1: Differentiating $f(x) = x^2$ from First Principles

Let's demonstrate the process with an example function:

$$
f(x) = x^2
$$

Step 1: Write the difference quotient:

$$
f'(a) = \lim_{h \to 0} \frac{(a+h)^2 - a^2}{h}
$$

Step 2: Expand $(a+h)^2$:

$$
(a+h)^2 = a^2 + 2ah + h^2
$$

Step 3: Substitute into the difference quotient:

$$
f'(a) = \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - a^2}{h}

\]

Step 4: Simplify numerator:

\[
f'(a) = \lim_{h \to 0} \frac{2ah + h^2}{h}
\]

Step 5: Factor \(h\) from numerator:

\[
f'(a) = \lim_{h \to 0} \frac{h(2a + h)}{h}
\]

Step 6: Cancel \(h\):

\[
f'(a) = \lim_{h \to 0} (2a + h)
\]

Step 7: Take the limit as \(h \to 0\):

\[
f'(a) = 2a
\]

Thus, the derivative of \(f(x) = x^2\) from first principles is:

\[
f'(x) = 2x
\]

Example 2: Differentiating \(f(x) = \sqrt{x}\) from First Principles

Let's now differentiate a square root function:

\[
f(x) = \sqrt{x}
\]

Step 1: Write the difference quotient:

\[
f'(a) = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h}
\]

Step 2: Rationalize the numerator to simplify:

Multiply numerator and denominator by the conjugate $\sqrt{a+h} + \sqrt{a}$:

$$
f'(a) = \lim_{h \to 0} \frac{\sqrt{a+h} - \sqrt{a}}{h} \times \frac{\sqrt{a+h} + \sqrt{a}}{\sqrt{a+h} + \sqrt{a}}
$$

Step 3: Simplify numerator using the difference of squares:

$$
\left(\sqrt{a+h}\right)^2 - \left(\sqrt{a}\right)^2 = (a+h) - a = h
$$

So,

$$
f'(a) = \lim_{h \to 0} \frac{h}{h (\sqrt{a+h} + \sqrt{a})}
$$

Step 4: Cancel h:

$$
f'(a) = \lim_{h \to 0} \frac{1}{\sqrt{a+h} + \sqrt{a}}
$$

Step 5: Take the limit as $h \to 0$:

$$
f'(a) = \frac{1}{2 \sqrt{a}}
$$

Therefore, the derivative of $f(x) = \sqrt{x}$ is:

$$
f'(x) = \frac{1}{2 \sqrt{x}}
$$

Example 3: Differentiating a Polynomial Function $f(x) = 3x^3 - 4x + 5$ from First Principles

Let's compute the derivative from first principles for a cubic function:

$$
f(x) = 3x^3 - 4x + 5
$$

\]

Step 1: Write the difference quotient:

\[
f'(a) = \lim_{h \to 0} \frac{3(a+h)^3 - 4(a+h) + 5 - [3a^3 - 4a + 5]}{h}
\]

Step 2: Expand \((a+h)^3\):

\[
(a+h)^3 = a^3 + 3a^2 h + 3a h^2 + h^3
\]

Step 3: Substitute into numerator:

\[
3(a^3 + 3a^2 h + 3a h^2 + h^3) - 4a - 4h + 5 - 3a^3 + 4a - 5
\]

Simplify numerator:

\[
3a^3 + 9a^2 h + 9a h^2 + 3h^3 - 4a - 4h + 5 - 3a^3 + 4a - 5
\]

Combine like terms:

- \(3a^3 - 3a^3 = 0\)
- \(-4a + 4a = 0\)
- \(5 - 5 = 0\)

Remaining terms:

\[
9a^2 h + 9a h^2 + 3h^3 - 4h
\]

Step 4: Write the difference quotient:

\[
f'(a) = \lim_{h \to 0} \frac{9a^2 h + 9a h^2 + 3h^3 - 4h}{h}
\]

Step 5: Factor \(h\) from numerator:

\[
f'(a) = \lim_{h \to 0} \frac{h(9a^2 + 9a h + 3h^2 - 4)}{h}
\]

Step 6: Cancel \(h\):

$$
f'(a) = \lim_{h \to 0} (9a^2 + 9a h + 3h^2 - 4)
$$

Step 7: Take the limit as $h \to 0$:

$$
f'(a) = 9a^2 - 4
$$

Since the derivative is a function of a, the derivative of $f(x)$ is:

$$
f'(x) = 9x^2 - 4
$$

Tips and Tricks for Computing Derivatives via First Principles

- Always simplify the numerator after expanding or rationalizing to make the limit process easier.
- Use conjugates when dealing with square roots to rationalize and simplify.
- Be patient with algebraic manipulations; meticulous expansion and factoring are key.
- Remember common binomial expansions, such as $(a+h)^n$, which can save time.
- Practice with various functions to understand different techniques and common patterns.

Comparison: First Principles vs. Shortcut Rules