Consider A = . Show That CA(x) =(xb)(xa)(x+a) And Find An Orthogonal Matrix P Such ThatP-1AP Is Diagonal.

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Introduction

In linear algebra, understanding the properties of matrices through their characteristic and minimal polynomials, eigenvalues, and eigenvectors is fundamental. When dealing with a specific matrix A, it is often essential to compute its characteristic polynomial $C_A(x)$, analyze its eigenstructure, and find a suitable basis of eigenvectors to diagonalize A via an orthogonal transformation. This process simplifies many matrix computations, allowing for straightforward powers, exponentials, and functional calculus.

In this article, we will explore a particular case where a matrix A is given or constructed such that its characteristic polynomial $C_A(x)$ can be expressed as a product of specific factors, namely $(x - b)$, $(x - a)$, and $(x + a)$. We will demonstrate how to derive this characteristic polynomial explicitly, verify the roots (eigenvalues), and find an orthogonal matrix P that diagonalizes A, i.e., $P^{-1}AP$ is a diagonal matrix containing the eigenvalues.

Understanding the Characteristic Polynomial

Definition and Significance

The characteristic polynomial $C_A(x)$ of an $n \times n$ matrix A is defined as:

$$
C_A(x) = \det(xI - A),
$$

where I is the identity matrix of appropriate size. The roots of $C_A(x)$ are precisely the eigenvalues of A. The algebraic multiplicities of these roots correspond to the multiplicities of eigenvalues.

Understanding $C_A(x)$ allows us to:

- Find eigenvalues analytically.
- Determine the diagonalizability of A.
- Construct eigenvectors for diagonalization.

Given Conditions and Form of $C_A(x)$

Suppose the matrix A has characteristic polynomial:

$$
C_A(x) = (x - b)(x - a)(x + a).
$$

This indicates that A has eigenvalues:

- $\lambda_1 = b$,
- $\lambda_2 = a$,
- $\lambda_3 = -a$.

Assuming these roots are distinct, A is diagonalizable over the real numbers, and the eigenvectors form a basis of eigenvectors.

Note: The form of $C_A(x)$ suggests that the matrix A is a 3x3 matrix, as it has three eigenvalues. It is crucial to verify whether the eigenvalues indeed correspond to the roots of $C_A(x)$, which depends on the specific entries of A.

Constructing the Matrix A and Deriving $C_A(x)$

Assuming A is a 3x3 matrix with specific entries

Let's consider a symmetric matrix A of the form:

$$
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{12} & a_{22} & a_{23} \\
a_{13} & a_{23} & a_{33}
\end{bmatrix}.
$$

Suppose A is constructed such that its characteristic polynomial factors as:

$$
C_A(x) = (x - b)(x - a)(x + a).
$$

This factorization indicates that the eigenvalues are real and symmetric about zero, with one eigenvalue possibly distinct (b).

Key points:

- The roots are real and distinct, ensuring diagonalizability.
- The eigenvalues are explicitly known.

Deriving the characteristic polynomial

The characteristic polynomial of A is:

$$
C_A(x) = \det(xI - A).
$$

Expanding this determinant yields a cubic polynomial:

$$
x^3 - \text{tr}(A) x^2 + \frac{1}{2} \left((\text{tr}(A))^2 - \text{tr}(A^2) \right) x - \det(A).
$$

Matching this with the factored form:

$$
(x - b)(x - a)(x + a),
$$

we obtain the following relations:

1. Sum of eigenvalues:

$$
b + a + (-a) = b,
$$

which is the trace of A:

$$
\text{tr}(A) = b.
$$

2. Sum of products of eigenvalues two at a time:

$$
b \cdot a + b \cdot (-a) + a \cdot (-a) = (ba - ba) - a^2 = -a^2.
$$

This sum relates to the coefficient of x.

3. The product of all eigenvalues:

$$
b \cdot a \cdot (-a) = -b a^2,
$$

which equals $\det(A)$.

Using these relations, we can identify the coefficients of the characteristic polynomial in terms of a and b, and vice versa.

Finding Eigenvectors and Diagonalization

Eigenvectors corresponding to each eigenvalue

To diagonalize A, we need to find eigenvectors v satisfying:

$$
A v = \lambda v,
$$

for each eigenvalue λ. The process involves solving the homogeneous system:

$$
(A - \lambda I)v = 0.
$$

Once eigenvectors are found, normalize them to have unit length to form an orthonormal basis.

Constructing the orthogonal matrix P

Let v_1, v_2, v_3 be orthonormal eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Define:

$$
P = [v_1 \quad v_2 \quad v_3],
$$

which is an orthogonal matrix $(P^T P = I)$.

The similarity transformation:

$$
P^{-1} A P = P^T A P,
$$

diagonalizes A:

$$
P^T A P = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3).
$$

Explicit Construction of P

Step-by-step process:

1. Compute eigenvectors:

- For each eigenvalue λ, solve $(A - \lambda I) v = 0$.
- Find a basis for the null space.

2. Normalize eigenvectors:

- Convert each eigenvector to a unit vector.
- Ensure the set of eigenvectors is orthogonal; if not, apply the Gram-Schmidt process.

3. Form matrix P:

- Arrange the orthonormal eigenvectors as columns.

4. Verify orthogonality:

- Confirm $P^T P = I$.

5. Diagonalize A:

- Compute $P^T A P$, which should be diagonal with eigenvalues on the diagonal.

Summary and Final Remarks

In this article, we've outlined the process of analyzing a matrix A with a characteristic polynomial $C_A(x) = (x - b)(x - a)(x + a)$. We demonstrated how the roots of $C_A(x)$ reveal the eigenvalues and how these eigenvalues guide us in constructing eigenvectors. The process of diagonalization involves forming an orthogonal matrix P composed of orthonormal eigenvectors, such that $P^{-1} A P$ is diagonal.

This approach not only simplifies computational tasks involving powers or functions of A but also reveals the spectral structure of A. The key steps involve:

- Factoring the characteristic polynomial.
- Solving for eigenvectors.
- Orthogonalizing eigenvectors.
- Assembling the matrix P.

By following these steps, one can effectively diagonalize matrices with known eigenvalues, especially in symmetric or orthogonally diagonalizable cases.

References

- Lay, David C. Linear Algebra and Its Applications. 4th Edition. Pearson, 2012.
- Strang, Gilbert. Introduction to Linear Algebra. 5th Edition. Wellesley-Cambridge Press, 2016.
- Hoffman, Kenneth, and Ray Kunze. Linear Algebra. 2nd Edition. Prentice-Hall, 1971.

Note: For specific numerical examples or matrices, substitute particular values into the outlined procedures to compute explicit eigenvectors and the orthogonal matrix \(P \).

Frequently Asked Questions

Given a matrix A, how can we verify that the characteristic polynomial is of the form CA(x) = (xb)(xa)(x + a)?

To verify this, we compute the characteristic polynomial CA(x) by calculating the determinant of (A - xI). If the resulting polynomial factors into (xb)(xa)(x + a), it indicates that the eigenvalues are 0 (with multiplicity related to x^b and x^a) and -a. Confirming the factorization matches the given form validates the characteristic polynomial.

How do we derive the form of CA(x) = (xb)(xa)(x + a) from the matrix A?

The form of CA(x) is derived by analyzing the eigenvalues of A through its characteristic polynomial. When A has eigenvalues 0 (with multiplicity b and a) and -a, the polynomial factors accordingly. This typically involves calculating det(A - xI) and factoring the resulting expression to reveal the (xb)(xa)(x + a) structure.

What is the procedure to find an orthogonal matrix P such that $P^{-1}AP$ is diagonal?

First, find all eigenvalues of A by solving CA(x) = 0. Then, determine the eigenvectors corresponding to each eigenvalue. Normalize these eigenvectors to form an orthonormal basis. Assemble these orthonormal eigenvectors into the matrix P. Since the eigenvectors are orthonormal, P will be an orthogonal matrix satisfying $P^{-1}AP = D$, where D is a diagonal matrix with eigenvalues on the diagonal.

Why is it important to find an orthogonal matrix P for diagonalization of A?

Using an orthogonal matrix P ensures numerical stability and preserves the inner product structure. It simplifies computations because P^{-1} equals P□ , making the process of diagonalization more straightforward and reliable, especially for symmetric matrices where orthogonal diagonalization guarantees real eigenvalues and orthogonal eigenvectors.

If A has eigenvalues 0 and -a, how does this influence the structure of the diagonal matrix after similarity transformation?

The diagonal matrix D will have the eigenvalues 0 and -a along its diagonal, with multiplicities corresponding to the algebraic multiplicities of the eigenvalues. Specifically, D will be a diagonal matrix with entries like [0, 0, ..., 0, -a, -a, ..., -a], reflecting the spectrum of A after the similarity transformation by the orthogonal matrix P.

Additional Resources

Consider A = . Show That CA(x) = (xb)(xa)(x + a) And Find An Orthogonal Matrix P Such That P-1AP Is Diagonal.

Introduction: Unveiling the Matrix and Its Characteristic Polynomial

Mathematics is often about uncovering relationships hidden within algebraic structures, especially matrices, which are fundamental in numerous scientific and engineering applications. This article explores a specific matrix, denoted as A, and guides you through two key tasks: first, to demonstrate a particular form of its characteristic polynomial, and second, to find an orthogonal matrix P that diagonalizes A. These are cornerstone concepts in linear algebra, essential for understanding eigenvalues, eigenvectors, and matrix diagonalization, all of which have practical implications in diverse fields such as quantum mechanics, control systems, and data analysis.

The problem is presented with a concise but intriguing prompt: "Consider A = . Show that CA(x) = (xb)(xa)(x + a) and find an orthogonal matrix P such that P-1AP is diagonal." While the notation is somewhat abstract, the core ideas revolve around characteristic polynomials and orthogonal diagonalization. Let's delve into the details, starting with understanding the matrix A and the characteristic polynomial.

Understanding the Matrix A and Its Characteristic Polynomial

Before tackling the algebraic manipulations, it's essential to clarify what the matrix A represents. Although the notation "A = ." suggests an incomplete reference, in typical linear algebra problems involving characteristic polynomials and diagonalization, A is often a real symmetric matrix or a matrix with specific properties that make such operations feasible.

For the purpose of this discussion, assume A is a 2x2 or 3x3 matrix with real entries, specified or implied by the context of the problem. The goal is to compute its characteristic polynomial, denoted as CA(x), which is defined as:

$$CA(x) = \det(xI - A)$$

This polynomial encapsulates the eigenvalues of A, which are the roots of CA(x). The problem hints at a particular factorization of CA(x):

$$CA(x) = (xb)(xa)(x + a)$$

or a similar form, indicating that the eigenvalues are related to parameters x, a, and possibly other variables. The notation suggests that the characteristic polynomial factors into linear factors, revealing the eigenvalues directly.

Factorization of the Characteristic Polynomial

Suppose A is such that its characteristic polynomial factors as:

$$CA(x) = (x - \lambda_1)(x - \lambda_2)(x - \lambda_3)$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of A. The expression given, $(xb)(xa)(x + a)$, suggests eigenvalues at:

- $x = 0$ (possibly with multiplicity related to b)
- $x = -a$

To match this, one might interpret the polynomial as:

$$ CA(x) = x^2 (x + a) $$

or with parameters, as:

$$ CA(x) = (x)(x + a)(x + b) $$

which implies eigenvalues at 0, $-a$, and $-b$.

In practice, to verify this, one computes the determinant $\det(xI - A)$ explicitly for the specific matrix A, then factors the resulting polynomial to identify eigenvalues.

Step-by-Step Calculation of the Characteristic Polynomial

Let's consider a concrete example of a matrix A to illustrate the process:

Suppose:

$$ A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & b & 0 \end{bmatrix} $$

This symmetric matrix models a system with parameters a and b. The characteristic polynomial is:

$$ CA(x) = \det(xI - A) $$

which computes as:

$$
\det \begin{bmatrix}
x - a & 0 & 0 \\
0 & x & -b \\
0 & -b & x
\end{bmatrix}
$$

Calculating this determinant:

$$
(x - a) \cdot \det \begin{bmatrix} x & -b \\ -b & x \end{bmatrix} = (x - a)(x^2 - b^2)
$$

Therefore,

$$ CA(x) = (x - a)(x - b)(x + b) $$

which factors into linear terms, indicating eigenvalues at a, b, and $-b$. This aligns with the general form suggested earlier.

Key Point: The factorization of the characteristic polynomial reveals the eigenvalues directly, which are essential for the diagonalization process.

Diagonalization of A Using Orthogonal Matrices

Having identified the eigenvalues, the next step is to find an orthogonal matrix P such that:

$$P^{-1} A P = D$$

where D is a diagonal matrix containing the eigenvalues. For real symmetric matrices, the Spectral Theorem guarantees the existence of an orthogonal matrix P that diagonalizes A. This is a powerful result, simplifying many matrix computations.

Step 1: Find Eigenvalues

From the characteristic polynomial, eigenvalues are known. For our example:

$$\lambda_1 = a, \quad \lambda_2 = b, \quad \lambda_3 = -b$$

Step 2: Find Eigenvectors

For each eigenvalue, solve:

$$(A - \lambda I) \mathbf{v} = 0$$

This involves solving linear systems:

- For $\lambda = a$:

$$(A - aI) \mathbf{v} = 0$$

- For $\lambda = b$:

$$(A - bI) \mathbf{v} = 0$$

- For $\lambda = -b$:

$$(A + bI) \mathbf{v} = 0$$

Eigenvectors are determined up to scalar multiples, and orthogonality is checked via the dot product.

Step 3: Orthonormalize Eigenvectors

Using the Gram-Schmidt process, or directly constructing orthogonal eigenvectors (possible since A is symmetric), normalize them to form the columns of P.

Step 4: Form the Matrix P

Arrange the orthonormal eigenvectors as columns in P:

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$$

This matrix is orthogonal ($P^T P = I$), and the similarity transformation:

$$
P^T A P = D
$$

yields a diagonal matrix with eigenvalues.

Practical Example: Constructing Orthogonal Diagonalization

Suppose the eigenvectors for the matrix A are:

$$
\mathbf{v}_a = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}
$$
$$
\mathbf{v}_b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$
$$
\mathbf{v}_{-b} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

These are orthonormal vectors corresponding to eigenvalues a, b, and $-b$. The matrix P is then:

$$
P = \begin{bmatrix}
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}
\end{bmatrix}
$$

and the diagonal matrix:

$$
D = \begin{bmatrix}
a & 0 & 0 \\
0 & b & 0 \\
0 & 0 & -b
\end{bmatrix}
$$

This process showcases how orthogonal matrices facilitate diagonalization, simplifying the analysis of matrix A's behavior.

Conclusion: The Power of Characteristic Polynomials and Orthogonal Diagonalization

This exploration underscores the vital interplay between characteristic polynomials, eigenvalues, eigenvectors, and the diagonalization of matrices, particularly through orthogonal transformations. Starting from the factorization of the characteristic polynomial, we can directly identify eigenvalues, then find orthogonal eigenvectors, and ultimately construct the orthogonal matrix P that diagonalizes A.

The methods described are not only theoretically elegant but also practically invaluable. Diagonalization simplifies complex matrix functions, enables efficient computations, and deepens our understanding of the underlying linear transformations. Whether in quantum physics, data science, or engineering, these tools are foundational for analyzing systems characterized by matrices.

By mastering these techniques, students and professionals alike can unlock the full potential of linear algebra, transforming abstract algebraic expressions into tangible insights about the structure and behavior of complex systems.

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