Consider The Function F(x) = 1 1 2ex, X 0, 0, X < 0. Show That F Is A Cumulative Distribution Function

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Introduction

In probability theory and statistics, understanding the properties of cumulative distribution functions (CDFs) is fundamental for analyzing random variables. A CDF provides the probability that a random variable X takes a value less than or equal to a specific point x. In this article, we will explore the function:

\[
F(x) =
\begin{cases}
0, & x < 0 \\
1 - \frac{1}{2} e^{x}, & x \geq 0
\end{cases}
\]

and demonstrate that it qualifies as a valid CDF. We will analyze its properties step by step, ensuring a comprehensive understanding of why this function adheres to the criteria of a CDF.

Understanding the Function \(F(x) \)

Before diving into the proof, it's essential to clearly understand the structure of the function:

- For \(x < 0 \), \(F(x) = 0 \).
- For \(x \geq 0 \), \(F(x) = 1 - \frac{1}{2} e^{x} \).

This piecewise definition suggests that the function is constant and zero for negative values of \(x \), and then transitions into an increasing function for non-negative \(x \).

Properties of a Valid Cumulative Distribution Function

A function $F(x)$ qualifies as a CDF if it satisfies the following properties:

1. Non-decreasing Nature

- $F(x)$ must be non-decreasing over the entire real line.

2. Right-Continuity

- $F(x)$ must be right-continuous at every point x.

3. Limits at Infinity

- $\lim_{x \to -\infty} F(x) = 0$.
- $\lim_{x \to +\infty} F(x) = 1$.

4. Boundedness

- $0 \leq F(x) \leq 1$ for all x.

Our goal is to verify these properties for the given piecewise function.

Step-by-Step Verification

Step 1: Verify Limits at Infinity

- At $x \to -\infty$:

Given $F(x) = 0$ for $x < 0$, and the function is constant (zero) for all $x < 0$. Therefore,

$$
\lim_{x \to -\infty} F(x) = 0
$$

- At $x \to +\infty$:

For large $x \geq 0$:

$$
F(x) = 1 - \frac{1}{2} e^{x}
$$

As $x \to +\infty$, $e^{x} \to +\infty$, so:

$$
F(x) \to 1 - \frac{1}{2} \times (+\infty) = -\infty
$$

This indicates that $F(x)$ tends to $-\infty$, which conflicts with the requirement that $F(x) \to 1$. However, this suggests a misinterpretation in the function's form. Based on typical CDF structures, perhaps the intended function is:

$$
F(x) =
\begin{cases}
0, & x < 0 \\
1 - \frac{1}{2} e^{-x}, & x \geq 0
\end{cases}
$$

or similar. Alternatively, perhaps the function is:

$$
F(x) = 1 - \frac{1}{2} e^{x} \quad \text{for } x \geq 0
$$

which is decreasing and not suitable as a CDF.

Important correction:

Given the function's form in the problem, perhaps the correct interpretation is:

$$
F(x) =
\begin{cases}
0, & x < 0 \\
1 - \frac{1}{2} e^{-x}, & x \geq 0
\end{cases}
$$

which is a typical form for a distribution function.

Corrected Function Analysis

Let's assume the function is:

$$
F(x) =
\begin{cases}
0, & x < 0 \\
1 - \frac{1}{2} e^{-x}, & x \geq 0
\end{cases}
$$

This form makes sense because:

- For $x \to +\infty$, $e^{-x} \to 0$, so $F(x) \to 1$.
- For $x \to 0^{+}$, $e^{0} = 1$, so $F(0) = 1 - \frac{1}{2} \times 1 = \frac{1}{2}$.
- For $x < 0$, $F(x) = 0$.

Now, we can proceed with verifying the properties with this corrected function.

Verifying the Corrected CDF

Property 1: Limits at Infinity

- $\lim_{x \to -\infty} F(x) = 0$. (since $F(x) = 0$ for $x < 0$)
- $\lim_{x \to +\infty} F(x) = 1 - 0 = 1$.

Conclusion: The limits align with the properties of a CDF.

Property 2: Non-decreasing Behavior

- For $x < 0$, $F(x) = 0$ (constant).
- For $x \geq 0$, $F(x) = 1 - \frac{1}{2} e^{-x}$.

Calculate the derivative for $x \geq 0$:

$$
F'(x) = \frac{1}{2} e^{-x} > 0
$$

since $e^{-x} > 0$ for all x, the derivative is positive, indicating $F(x)$ is strictly increasing on $[0, \infty)$.

At $x = 0$, the function is continuous:

$$
F(0) = 1 - \frac{1}{2} \times 1 = \frac{1}{2}
$$

and for $x < 0$, $F(x) = 0$, so the function jumps from 0 to 0.5 at $x=0$. The jump is acceptable for a CDF (a jump corresponds to a point mass).

Conclusion: $F(x)$ is non-decreasing, with a jump at $x=0$.

Property 3: Right-Continuity

- For $x < 0$, $F(x)$ is constant and continuous.
- At $x=0$:

$$
\lim_{x \to 0^{+}} F(x) = 1 - \frac{1}{2} e^{0} = 1 - \frac{1}{2} = \frac{1}{2}
$$
$$
F(0) = \frac{1}{2}
$$

so F is right-continuous at 0.

- For $x > 0$, $F(x)$ is continuous.

Conclusion: The function $F(x)$ is right-continuous everywhere.

Property 4: Bounds

- For $x<0$, $F(x) = 0$.
- For $x \geq 0$:

$$
F(x) = 1 - \frac{1}{2} e^{-x}
$$

Since $e^{-x} > 0$, and for $x \geq 0$, $e^{-x} \leq 1$, so:

$$
F(x) \geq 1 - \frac{1}{2} \times 1 = \frac{1}{2}
$$

and

\[
F(x) \leq 1
\]

But, at \(x=0 \), \(F(0)=0.5 \), and as \(x \to +\infty \), \(F(x) \to 1 \). Therefore, \(0 \leq F(x) \leq 1 \) for all \(x \).

Summary of Findings

- Limits at infinity: \(\lim_{x \to -\infty} F(x) = 0 \), \(\lim_{x \to +\infty} F(x) = 1 \).
- Non-decreasing: \(F(x) \) is constant for \(x<0 \), increasing for \(x \geq 0 \).
- Right-continuous: Continuous everywhere, including at the jump at \(x=0 \).
- Bounded: Values between 0 and 1.

Therefore, \(F(x) \) satisfies all the properties of a cumulative distribution function.

Interpretation and Significance

Understanding the

Frequently Asked Questions

What are the key properties that a function must satisfy to be considered a cumulative distribution function (CDF)?

A function must be non-decreasing, right-continuous, satisfy the limits $F(x)\to 0$ as $x\to -\infty$, and $F(x)\to 1$ as $x\to\infty$ to qualify as a CDF.

Given the function $F(x) = 1/2\ e^x$ for $x < 0$ and $F(x) = 1$ for $x \geq 0$, how do we verify it is a valid CDF?

We verify by checking that $F(x)$ is non-decreasing, right-continuous, approaches 0 as $x\to -\infty$, and approaches 1 as $x\to\infty$. The function satisfies these properties, confirming it is a CDF.

Why is the function F(x) = 1/2 e^x for x < 0 considered valid as part of a CDF?

Because it is non-decreasing on the real line, right-continuous, approaches 0 as x→-∞, and approaches 1/2 as x approaches 0 from the left, consistent with the properties of a CDF.

How does the jump at x=0 in the function F(x) affect its status as a CDF?

The jump at x=0 (from 1/2 e^0 = 1/2 to 1) indicates a discontinuity, which is allowed in CDFs as long as the function is right-continuous. This jump represents a point mass in the distribution.

What is the significance of the limit F(x)→1 as x→∞ in confirming F(x) as a CDF?

This limit confirms that the total probability sums to 1, fulfilling a key requirement of probability distributions and validating F(x) as a cumulative distribution function.

Can F(x) be considered a valid CDF if it is not continuous everywhere? Why or why not?

Yes, F(x) can have discontinuities at points, representing discrete jumps, and still be a valid CDF. Continuity is not required everywhere; right-continuity is sufficient.

How would you interpret the probability distribution described by the given F(x)?

The distribution has a continuous part for x<0, with a density proportional to e^x, and a point mass at x=0, since F(x) jumps there. This indicates a mixed distribution with discrete and continuous components.

Additional Resources

Consider The Function F(x) = \(\left\{\begin{array}{ll} 1 & x \geq 0 \\ 1/2 e^x & x < 0 \end{array}\): A Detailed Investigation into Its Role as a Cumulative Distribution Function

In the realm of probability theory and statistical analysis, the characterization and validation of distribution functions are of paramount importance. The function \(F(x) \), defined piecewise as

\[
F(x) =
\begin{cases}

\frac{1}{2} e^{x}, & x < 0 \\
1, & x \geq 0
\end{cases}
\]

presents an intriguing candidate for a cumulative distribution function (CDF). The question posed is: "Consider The Function F(x) = \(\left\{\begin{array}{ll} 1 & x \geq 0 \\ 1/2 e^x & x < 0 \end{array}\): Show That F Is A Cumulative Distribution Function."

This article aims to rigorously analyze this function, demonstrating that it indeed qualifies as a legitimate CDF by examining its properties, underlying structure, and probabilistic implications. We undertake a thorough, step-by-step investigation, integrating foundational principles with detailed proofs and illustrative insights.

Understanding the Foundations: What Constitutes a Cumulative Distribution Function?

Before delving into the specifics of \(F(x) \), it is crucial to recall the defining properties of a CDF:

Fundamental Properties of a CDF

A function \(F: \mathbb{R} \to [0, 1] \) is a CDF if and only if it satisfies the following:

1. Non-decreasing: \(F(x) \) is non-decreasing; that is, \(x_1 \leq x_2 \Rightarrow F(x_1) \leq F(x_2) \).

2. Right-continuity: \(F(x) \) is right-continuous for all \(x \in \mathbb{R} \).

3. Limits at infinities:
- \(\lim_{x \to -\infty} F(x) = 0 \).
- \(\lim_{x \to +\infty} F(x) = 1 \).

4. Range: \(0 \leq F(x) \leq 1 \) for all \(x \).

Our task is to verify that the given piecewise function \(F \) adheres to these properties, establishing it as a valid CDF.

Dissecting the Piecewise Function \(F(x) \)

The function:

$$
F(x) =
\begin{cases}
\frac{1}{2} e^{x}, & x < 0 \\
1, & x \geq 0
\end{cases}
$$

is designed with a simple structure that exhibits exponential behavior for negative x and plateaus at 1 for $x \geq 0$.

Visual Intuition

- For $x < 0$, $F(x)$ increases exponentially from 0 (at $x \to -\infty$) up to $\frac{1}{2} e^{0} = \frac{1}{2}$ (at $x \to 0^-$).
- At $x = 0$, $F(x)$ jumps to 1.
- For $x > 0$, $F(x)$ remains constant at 1.

This structure suggests a distribution that is concentrated on the negative real line with a particular mass at zero.

Verifying the Essential Properties

1. Range and Limits at Infinity

- At $x \to -\infty$:

Since $e^{x} \to 0$ as $x \to -\infty$, then

$$
F(x) \to \frac{1}{2} \times 0 = 0,
$$

satisfying $\lim_{x \to -\infty} F(x) = 0$.

- At $x \to +\infty$:

For $x \geq 0$, $F(x) = 1$, so

$$
\lim_{x \to +\infty} F(x) = 1,

\]

satisfying the second limit condition.

- Range:

For $x < 0$, $F(x) = \frac{1}{2} e^{x} \in (0, \frac{1}{2})$.

For $x \geq 0$, $F(x) = 1$.

Since $\frac{1}{2} e^{x}$ is always in $(0, \frac{1}{2})$ for $x<0$, and the function jumps to 1 at zero, the entire range of F is within $[0, 1]$.

2. Non-decreasing Behavior

- For $x < 0$:

$F(x) = \frac{1}{2} e^{x}$, which is strictly increasing as e^{x} is increasing in x.

- At $x=0$:

$F(0) = 1$.

- For $x > 0$:

$F(x) = 1$, constant.

- At the boundary $x \to 0^-$:

$F(0^-) = \frac{1}{2} e^{0} = \frac{1}{2}$.

Since $F(0) = 1$ and $F(0^-) = \frac{1}{2}$, the function exhibits a jump discontinuity at 0.

- Conclusion:

The function is non-decreasing overall, with an jump from $\frac{1}{2}$ to 1 at zero, satisfying the monotonicity property.

3. Right-Continuity

- For $x < 0$, F is continuous due to the exponential function.

- At $x=0$:

$$
F(0) = 1,
$$

and the limit from the left:

$$
\lim_{x \to 0^-} F(x) = \frac{1}{2} e^{0} = \frac{1}{2}.
$$

Since $F(0) = 1 \neq \frac{1}{2}$, F has a jump discontinuity at 0, which is permitted in a CDF because right-continuity only requires the function to be continuous from the right.

- For $x > 0$, F is constant at 1, thus continuous from the right.

Therefore, F is right-continuous everywhere, satisfying the right-continuity property.

Probabilistic Interpretation: Is F a Valid CDF?

Having verified the properties, the next step is to interpret F in probabilistic terms and confirm that it corresponds to a genuine distribution.

Total Probability Mass

- The change in F at $x=0$ is:

$$
F(0) - \lim_{x \to 0^-} F(x) = 1 - \frac{1}{2} = \frac{1}{2}.
$$

This indicates a mass of $\frac{1}{2}$ at $x=0$.

- For $x<0$, $F(x)$ is continuous and increasing, with its maximum approaching $\frac{1}{2}$ as $x \to 0^-$, suggesting that the distribution has support on $(-\infty, 0)$ with a "density" related to $\frac{1}{2} e^x$.

- For $x>0$, F remains constant at 1, confirming total probability sums to 1.

Formal Confirmation via Distribution Function Decomposition

The function F can be viewed as a mixture of:

- A continuous component on \((-\infty, 0) \),
- A point mass at \(x=0 \).

Specifically, the distribution \(P \) associated with \(F \) is characterized by:

\[
P((-\infty, x]) = F(x),
\]

and the corresponding probability measure \(\mu \) can be decomposed as:

\[
\mu = \underbrace{\left(\frac{1}{2} \cdot \text{Exponential}(1) \right)}_{\text{on } (-\infty, 0)} + \left(\frac{1}{2} \delta_0 \right),
\]

where:

- The continuous part has density \(f(x) = \frac{1}{2} e^{x} \) for \(x < 0 \),
- The discrete part assigns probability \(\frac{1}{2} \) at \(x=0 \),
- Total probability sums to

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