Determine The Output Of The Following Signal After Passingthrough An Ideal Low-pass Filter With A Cut-off

Determine The Output Of The Following Signal After Passingthrough An Ideal Low-pass Filter With A Cut-off

When analyzing signals in engineering, understanding how they behave when passing through various filters is essential. One common filter used in signal processing is the ideal low-pass filter, which allows signals below a certain cutoff frequency to pass through unaltered while attenuating frequencies above this threshold. In this article, we will explore how to determine the output of a given signal after it passes through an ideal low-pass filter with a specified cutoff frequency. By delving into the principles of filtering, the mathematical foundation, and practical examples, you'll gain a comprehensive understanding of this fundamental concept in signal processing.

Understanding the Ideal Low-pass Filter

What Is an Ideal Low-pass Filter?

An ideal low-pass filter is a hypothetical filter characterized by a perfect cutoff frequency (fc). It completely passes all frequency components below fc without any attenuation and completely blocks all components above fc. Its frequency response, H(f), is defined as:

- H(f) = 1 for |f| ≤ fc
- H(f) = 0 for |f| > fc

This sharp transition at the cutoff frequency makes it an idealized model, as real-world filters exhibit gradual roll-off.

Frequency Response and Its Significance

The frequency response determines how different frequency components of a signal are affected during filtering. For an ideal low-pass filter:

- Frequencies below fc remain unchanged.
- Frequencies above fc are completely eliminated.

This characteristic simplifies analysis, especially when working with signals in the frequency domain.

Analyzing the Input Signal

Types of Input Signals

The nature of the input signal greatly influences the output after filtering. Common types include:
- Sinusoidal signals
- Sum of multiple sinusoids
- Complex waveforms (e.g., square, sawtooth)
- Random signals or noise

Understanding the spectral composition of your input signal is critical for predicting the filtered output.

Fourier Transform of the Input Signal

To analyze the effect of the filter, it is often practical to work in the frequency domain:
- Compute the Fourier Transform of the input signal to obtain its spectrum.
- Identify the frequency components relative to the cutoff frequency.

For example, consider a simple sinusoid:
- $x(t) = A \sin(2\pi f0\ t)$
- Its Fourier Transform is two delta functions at $\pm f0$.

For more complex signals, the Fourier Transform provides a spectrum indicating the amplitude of each frequency component.

Applying the Ideal Low-pass Filter

Frequency Domain Multiplication

The filtering process in the frequency domain involves:
- Multiplying the Fourier Transform of the input signal by the filter's frequency response H(f).

Mathematically:
- $Y(f) = X(f) H(f)$

Where:
- $X(f)$ is the Fourier Transform of the input signal.
- $H(f)$ is the filter's frequency response.
- $Y(f)$ is the Fourier Transform of the output signal.

Because $H(f)$ is 1 for $|f| \leq fc$ and 0 elsewhere:
- Frequencies within the cutoff pass through unchanged.
- Frequencies outside are eliminated.

Inverse Fourier Transform to Obtain the Output Signal

After multiplication:
- Compute the inverse Fourier Transform of $Y(f)$ to obtain the time-domain output signal $y(t)$.

This process effectively "filters out" the high-frequency components, leaving only the desired portion of the spectrum.

Step-by-Step Procedure to Determine the Output

Step 1: Identify the Input Signal Spectrum

- Express the input signal mathematically.
- Calculate or determine its Fourier Transform.

Step 2: Define the Filter's Frequency Response

- For an ideal low-pass filter with cutoff frequency fc:
- $H(f) = 1$ for $|f| \leq fc$
- $H(f) = 0$ for $|f| > fc$

Step 3: Perform Frequency Domain Multiplication

- Multiply each component of $X(f)$ by $H(f)$:
- For frequencies within the cutoff, pass unchanged.
- For frequencies beyond the cutoff, set to zero.

Step 4: Calculate the Inverse Fourier Transform

- Convert the filtered spectrum back to the time domain.
- The resulting $y(t)$ is the output signal after filtering.

Practical Examples

Example 1: Sinusoidal Input Signal

Suppose the input is a pure sine wave:
- $x(t) = A \sin(2\pi f_0 t)$

Its spectrum:
- $X(f)$ consists of two delta functions at $\pm f_0$.

If:
- $f_c > f_0$, then the entire spectrum passes through:
- Output $y(t) = x(t)$

If:
- $f_c < f_0$, then the spectrum at $\pm f_0$ is eliminated:
- Output $y(t) = 0$ (no output)

Example 2: Sum of Sinusoids

Input:
- $x(t) = A_1 \sin(2\pi f_1 t) + A_2 \sin(2\pi f_2 t)$

Spectrum:
- Components at f_1 and f_2.

If:
- $f_c > \max(f_1, f_2)$, both pass through, and the output is identical to the input.
- f_c between f_1 and f_2, then only the components at f_1 pass, and f_2 is attenuated.
- $f_c < \min(f_1, f_2)$, both are blocked, resulting in zero output.

Example 3: Complex Signals

For signals like square waves:
- Their spectra contain multiple harmonics.

- Filtering with an ideal low-pass filter with a cutoff frequency below the harmonic frequencies results in a smoothed, sinusoidal-like output.

Common Considerations and Limitations

Real-World vs. Ideal Filters

While the ideal low-pass filter provides a clear-cut analysis, actual filters have:
- Gradual roll-off
- Non-zero attenuation above cutoff
- Phase distortions

These factors make the real-world filter response more complex.

Effect of Filter on Signal Integrity

- Sharp filters can introduce ringing artifacts (Gibbs phenomenon).
- Filters with gradual roll-off preserve more of the signal's shape.

Summary and Key Takeaways

- Determining the output of a signal after passing through an ideal low-pass filter involves analyzing the signal's spectrum.
- In the frequency domain, multiplication by the filter's response simplifies the process.
- The inverse Fourier Transform yields the time-domain output.
- The cutoff frequency critically influences which spectral components are retained or eliminated.

Conclusion

Understanding how to determine the output of a signal after passing through an ideal low-pass filter with a specified cutoff frequency is fundamental in signal processing. By analyzing the spectral components of the input signal, applying the filter's frequency response, and performing inverse transforms, engineers and technicians can predict and manipulate signals effectively. Whether in communications, audio processing, or control systems, mastering this process enables better design and analysis of filtering systems, ensuring signals are processed accurately and efficiently.

If you want to analyze a specific signal and cutoff frequency, following these steps provides a systematic approach to accurately determine the filtered output.

Frequently Asked Questions

What is the primary effect of an ideal low-pass filter with a given cutoff frequency on a signal?

An ideal low-pass filter allows frequencies below the cutoff to pass unchanged while completely attenuating frequencies above it, resulting in a smoothed or less high-frequency content signal.

How can the output of a signal after passing through an ideal low-pass filter be determined analytically?

By taking the Fourier transform of the input signal, multiplying it by the ideal low-pass filter's frequency response (a rectangular function), and then applying the inverse Fourier transform to obtain the time-domain output.

If the input signal contains frequency components higher than the cutoff frequency, what will be the effect on the output after filtering?

The high-frequency components above the cutoff will be completely attenuated, resulting in a smoother signal with reduced high-frequency content, potentially causing some loss of sharpness or detail.

Given a sinusoidal input signal with a frequency below the cutoff, what is the expected output after passing through an ideal low-pass filter?

The output will be the same as the input sinusoid, possibly with a phase shift, since frequencies below the cutoff pass through unaffected.

How does the cutoff frequency of the ideal low-pass filter influence the duration and shape of the filtered signal in the time domain?

A lower cutoff frequency results in more aggressive filtering, causing the output to be smoother and potentially more distorted in shape, while a higher cutoff preserves more of the original signal's details. The cutoff determines the balance between noise reduction and fidelity.

Additional Resources

Determine The Output Of The Following Signal After Passing Through An Ideal Low-pass Filter With A Cut-off is a fundamental problem in signal processing that illustrates how filters modify signals in both the time and frequency domains. Understanding this process is essential for engineers and students working on communications, audio processing, and systems design. When a signal passes through an ideal low-pass filter, the filter's primary function is to attenuate frequency components above a specified cutoff frequency while allowing those below it to pass unchanged. This article aims to provide a comprehensive overview of the principles involved, the mathematical foundation, and practical considerations to determine the output of such systems.

Introduction to Low-pass Filters

A low-pass filter (LPF) is a system that permits signals with frequencies below a certain cutoff frequency, f_c, to pass through with minimal attenuation, while significantly reducing the amplitude of higher-frequency signals. In the ideal case, the filter perfectly transmits all frequencies below f_c and completely blocks those above. This idealization simplifies the analysis but also introduces certain limitations and considerations in real-world applications.

Definition and Types of Low-pass Filters

Low-pass filters can be categorized into several types based on their implementation:
- Passive filters: Comprising resistors, capacitors, and inductors.
- Active filters: Using operational amplifiers along with passive components.
- Digital filters: Implemented via algorithms in digital signal processing systems.

However, the focus here is on the ideal low-pass filter, which can be considered a theoretical construct with a perfect rectangular frequency response.

Mathematical Representation of the Ideal Low-pass Filter

The ideal low-pass filter's transfer function, $H(f)$, is characterized as:

$$
H(f) = \begin{cases}
1, & |f| \leq f_c \\
0, & |f| > f_c
\end{cases}
$$

This step function in the frequency domain results in a rectangular filter response, which is simple to analyze but introduces certain artifacts in the time domain, such as ringing effects.

Frequency Domain Analysis

Given an input signal $x(t)$, its Fourier Transform $X(f)$ describes its frequency content:

$$
X(f) = \mathcal{F}\{x(t)\}
$$

The output in the frequency domain after passing through the ideal LPF is:

$$
Y(f) = H(f) \cdot X(f)
$$

which leads to the inverse Fourier Transform:

$$
y(t) = \mathcal{F}^{-1}\{Y(f)\}
$$

Because $H(f)$ is a rectangular function, the inverse transform involves the convolution of the input signal with the inverse Fourier transform of $H(f)$.

Time Domain Implication: Convolution with the Sinc Function

The inverse Fourier transform of the ideal low-pass filter's frequency response is a sinc function:

$$
h(t) = 2f_c \cdot \text{sinc}(2\pi f_c t)
$$

where:

$$
\text{sinc}(x) = \frac{\sin x}{x}
$$

This impulse response implies that the output signal in the time domain is the convolution of the input with this sinc function:

$$
y(t) = x(t) h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) \, d\tau
$$

Implication: Passing a signal through an ideal low-pass filter results in a smoothed version that is a weighted average of the original signal, effectively removing high-frequency components.

Determining the Output Signal: Step-by-Step Approach

1. Identify the Input Signal's Frequency Content

The first step involves analyzing $x(t)$ to find its Fourier transform $X(f)$. This may involve:
- Analytical calculation if $x(t)$ is known explicitly.
- Using Fourier series expansion for periodic signals.
- Employing Fourier analysis for complex signals.

Example: If $x(t)$ is a sum of sinusoids, its spectrum consists of discrete lines at their respective frequencies.

2. Apply the Filter's Frequency Response

Multiply $X(f)$ by $H(f)$:

$$
Y(f) = X(f) \quad \text{for} \ |f| \leq f_c
$$

and zero elsewhere. This step effectively removes the spectral components above f_c.

Practical note: For signals with spectral components above f_c, these will be completely eliminated.

3. Perform the Inverse Fourier Transform

Calculate:

$$
y(t) = \mathcal{F}^{-1}\{Y(f)\}
$$

which, because $Y(f)$ is the product of $X(f)$ and $H(f)$, corresponds to the convolution of $x(t)$ with $h(t)$.

4. Time Domain Interpretation

The result is a smoothed or filtered version of the original signal, with high-frequency details suppressed.

Practical Examples and Applications

Understanding the output involves applying the above steps to specific input signals. Let's consider some common cases:

Example 1: Sinusoidal Input

Suppose $x(t) = \sin(2\pi f_0 t)$, where $f_0 < f_c$. The Fourier transform has a spectral line at f_0. Since $f_0 \leq f_c$, the filter passes this component unchanged, and the output remains:

$$
y(t) = \sin(2\pi f_0 t)
$$

If $f_0 > f_c$, the component is removed, and the output is zero (assuming no other components).

Example 2: Square Wave

A square wave can be expressed as a sum of odd harmonics:

$$
x(t) = \frac{4}{\pi} \sum_{k=1,3,5,\dots}^\infty \frac{1}{k} \sin(2\pi k f_0 t)
$$

Passing through the low-pass filter with cutoff f_c retains only the fundamental and possibly some lower harmonics, resulting in a smoothed, rounded signal.

Features and Limitations of Ideal Low-pass Filters

Features:
- Perfect cutoff: Sharp transition at f_c, no transition band.
- Maximum attenuation: Completely blocks frequencies above f_c.
- Mathematically simple: Facilitates theoretical analysis.

Limitations:
- Non-physical: No real filter can have an infinitely sharp cutoff.
- Gibbs phenomenon: The abrupt cutoff causes ripples and ringing in the time domain.
- Infinite duration impulse response: The sinc function extends infinitely in time, making it impractical for real implementations.

Practical Considerations and Real-World Filters

In practice, filters are designed with a transition band rather than an abrupt cutoff, and their frequency response is smooth rather than rectangular. Nonetheless, analyzing the ideal case provides insight into the fundamental limits and behavior of filtering processes.

Pros of Ideal Filters:
- Clear understanding of the maximum possible filtering effect.

- Useful as a benchmark for real filter performance.

Cons:
- Not realizable with perfect sharpness.
- Induces ringing artifacts (Gibbs phenomenon).
- Infinite impulse response makes implementation impossible.

Summary and Conclusions

Determining the output of a signal after passing through an ideal low-pass filter involves understanding the filter's frequency response, applying Fourier analysis, and performing inverse transforms. The key steps include analyzing the input's spectrum, multiplying by the filter's transfer function, and transforming back to the time domain. The convolution in the time domain with a sinc function highlights the fundamental relationship between frequency filtering and time-domain smoothing.

While ideal filters are theoretical constructs, they serve as important tools for understanding the limitations and capabilities of real-world systems. Their analysis reveals critical insights such as the trade-off between filter sharpness and time-domain artifacts like ringing. Engineers can use these principles to design practical filters that approximate ideal behavior within physical and technological constraints.

In summary, the process of determining the output involves:
- Fourier analysis of the input.
- Applying the filter's frequency response.
- Inverse Fourier transformation to obtain the filtered signal.
- Interpreting the resulting smoothed or modified signal.

This comprehensive understanding forms a cornerstone of modern signal processing, enabling effective design and analysis of systems across communications, audio engineering, and beyond.

Determine The Output Of The Following Signal After Passingthrough An Ideal Low Pass Filter With A Cut Off

Determine The Output Of The Following Signal After Passingthrough An Ideal Low Pass Filter With A Cut Off

Related Articles

- A 45-year-old Male Presents With Persistent, Severe Stomach Pain. Testing Reveals A Peptic Ulcer. Further
- Find The Volume When The Region Between = 2sin() 1 And The X-axis For 6 5 6 Is Revolved About The Y-axis.

- A Patient Who Has Acute Pancreatitis Is Prescribed Famotidine. The Nurse Explains That This Drug Is Given

Back to Home