Determine Whether The Following Improper Integral Converges Or Diverges. If It Converges, Find Its Value.

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Introduction to Improper Integrals

Improper integrals are a class of integrals where the interval of integration is unbounded (infinite limits) or the integrand becomes unbounded (approaching infinity) at some point within the interval. These integrals are fundamental in advanced calculus, physics, engineering, and various applied sciences, as they often model real-world phenomena such as probabilities, decay processes, and potential fields.

Understanding whether an improper integral converges or diverges is crucial because convergence implies the integral approaches a finite value, making it meaningful in practical applications. Conversely, divergence indicates the integral does not settle to a finite number, often signaling an infinite or undefined quantity.

In this article, we will explore methods to analyze the convergence or divergence of improper integrals and, when possible, determine their exact values. We will use various example integrals to illustrate these techniques.

Types of Improper Integrals

1. Integrals with Infinite Limits

These are integrals where the interval extends to infinity or negative infinity:

- $\int_a^{\infty} f(x) \, dx$
- $\int_{-\infty}^b f(x) \, dx$

- $\int_{-\infty}^{\infty} f(x) \, dx$

2. Integrals with Unbounded Integrand

These integrals involve functions that become unbounded at some point within the interval:

- $\int_a^b f(x) \, dx$, where $f(x)$ approaches infinity at $x=a$ or $x=b$.

Methods for Determining Convergence or Divergence

To analyze improper integrals, mathematicians employ several strategies:

1. Comparison Test

- Principle: If $0 \leq f(x) \leq g(x)$ for all x in the interval, then:

- If $\int_a^\infty g(x) \, dx$ converges, so does $\int_a^\infty f(x) \, dx$.
- If $\int_a^\infty f(x) \, dx$ diverges, so does $\int_a^\infty g(x) \, dx$.

- Application: Useful when the integrand can be compared to a known convergent or divergent integral.

2. Limit Comparison Test

- Principle: For $f(x), g(x) > 0$, if

$$
\lim_{x \to \infty} \frac{f(x)}{g(x)} = c
$$

where c is a finite positive constant, then both integrals $\int_a^\infty f(x) \, dx$ and $\int_a^\infty g(x) \, dx$ either both converge or both diverge.

3. p-Integral Test

- Principle: For integrals of the form $\int_a^\infty \frac{1}{x^p} \, dx$:

- Converges if $p > 1$.
- Diverges if $p \leq 1$.

4. Direct Computation

- Find an antiderivative $F(x)$ such that $F'(x) = f(x)$.
- Evaluate the limit:

$$
\lim_{t \to \infty} [F(t) - F(a)] \quad \text{or} \quad \lim_{a \to -\infty} [F(b) - F(a)]
$$

- If the limit exists and is finite, the integral converges; otherwise, it diverges.

Worked Examples and Analysis

Example 1: Convergence of $\int_1^{\infty} \frac{1}{x^p} \, dx$

Step 1: Recognize the form

This is a p-integral, a classic example.

Step 2: Find the antiderivative

$$
\int \frac{1}{x^p} dx =
\begin{cases}
\frac{x^{1 - p}}{1 - p} + C, & p \neq 1 \\
\ln|x| + C, & p=1
\end{cases}
$$

Step 3: Evaluate the limit

- For $p \neq 1$:

$$
\int_1^{\infty} \frac{1}{x^p} dx = \lim_{t \to \infty} \left[\frac{t^{1 - p}}{1 - p} - \frac{1^{1 - p}}{1 - p}\right]
$$

- Case 1: $p > 1$

$$
1 - p < 0 \implies t^{1 - p} \to 0 \quad \text{as } t \to \infty
$$

Thus,

$$
\int_1^{\infty} \frac{1}{x^p} dx = \frac{0}{1 - p} - \frac{1}{1 - p} = -\frac{1}{1 - p}
$$

which is finite, indicating convergence.

- Case 2: $p \leq 1$

$$
t^{1 - p} \to \infty \quad \text{or remains finite for } p=1
$$

Specifically,

- If $p=1$:

$$
\int_1^{\infty} \frac{1}{x} dx = \lim_{t \to \infty} \ln t = \infty
$$

diverges.

- If $p < 1$:

$$
1 - p > 0 \implies t^{1 - p} \to \infty
$$

diverges.

Conclusion:

- The improper integral converges if and only if $p > 1$.

- Its value when convergent:

\[

\int_1^{\infty} \frac{1}{x^p} dx = \frac{1}{p - 1}
\]

Example 2: Integral with Unbounded Integrand \(\int_0^{1} \frac{1}{x^{\alpha}} dx\)

Step 1: Recognize the form

This integral has an unbounded integrand at \(x=0\).

Step 2: Find the antiderivative

\[
\int \frac{1}{x^{\alpha}} dx =
\begin{cases}
\frac{x^{1 - \alpha}}{1 - \alpha} + C, & \alpha \neq 1 \\
\ln|x| + C, & \alpha=1
\end{cases}
\]

Step 3: Evaluate the improper integral

\[
\int_0^{1} \frac{1}{x^{\alpha}} dx = \lim_{\varepsilon \to 0^+}
\int_\varepsilon^{1} x^{-\alpha} dx
\]

- For \(\alpha \neq 1\):

\[
= \lim_{\varepsilon \to 0^+} \left[\frac{x^{1 - \alpha}}{1 - \alpha}
\right]_\varepsilon^{1} = \frac{1^{1 - \alpha} - \varepsilon^{1 - \alpha}}{1
- \alpha}
\]

- As \(\varepsilon \to 0^+\):

- If \(1 - \alpha > 0 \Rightarrow \alpha < 1\):

\[
\varepsilon^{1 - \alpha} \to 0
\]
so the integral converges to \(\frac{1}{1 - \alpha}\).

- If \(1 - \alpha < 0 \Rightarrow \alpha > 1\):

\[

\varepsilon^{1 - \alpha} \to \infty
\]
so the integral diverges.

- For \(\alpha=1\):

\[
\int_\varepsilon^1 \frac{1}{x} dx = \ln 1 - \ln \varepsilon = - \ln \varepsilon
\]
which tends to infinity as \(\varepsilon \to 0^+\), so diverges.

Conclusion:

- The integral converges if and only if \(\alpha < 1\).

- Its value when convergent:

\[
\int_0^{1} \frac{1}{x^{\alpha}} dx = \frac{1}{1 - \alpha}
\]

for \(\alpha < 1\).

Summary of Convergence Criteria

Type of integral	Condition for convergence	Notes
\(\int_a^{\infty} \frac{1}{x^p} dx\)	\(p > 1

Frequently Asked Questions

How do I determine whether the improper integral $\int_1^\infty$ 1/(x^2) dx converges or diverges?

Since the integrand 1/x^2 is continuous on [1, ∞) and behaves like a p-integral with p=2>1, the integral converges. Its value is $\int_1^\infty$ 1/x^2 dx = [-1/x $]_1^\infty$ = 0 - (-1/1) = 1.

Does the improper integral $\int_0^1 1/\sqrt{x}\, dx$ converge or diverge?

The integrand $1/\sqrt{x}$ has a vertical asymptote at $x=0$. Since $\int_0^1 1/\sqrt{x}\, dx = 2\sqrt{x}$ evaluated from 0 to 1, which equals $2(1) - 2(0) = 2$, the integral converges and its value is 2.

How can I test if the improper integral $\int_1^\infty e^{-x}\, dx$ converges and find its value?

The exponential decay e^{-x} decreases rapidly, so the integral converges. Its value is $\int_1^\infty e^{-x}\, dx = [-e^{-x}]_1^\infty = 0 - (-e^{-1}) = 1/e$.

What is the method to determine if $\int_0^\infty (1/x)\, dx$ diverges or converges?

Since $\int_0^\infty 1/x\, dx$ is an improper integral with a divergence at 0 and infinity, splitting it into $\int_0^1 1/x\, dx$ and $\int_1^\infty 1/x\, dx$ shows both diverge (the first diverges to infinity at 0, the second diverges at infinity). Therefore, the integral diverges.

Does the improper integral $\int_2^\infty 1/(x \ln x)\, dx$ converge or diverge?

This integral diverges. It is known that $\int_2^\infty 1/(x \ln x)\, dx = \ln(\ln x)$ evaluated from 2 to ∞, which tends to infinity as x approaches infinity, so the integral diverges.

How do I evaluate the improper integral $\int_0^1 x^{p-1}\, dx$, and when does it converge?

This is a p-integral. It converges when $p > 0$, and its value is $1/p$. For example, if $p=1/2$, the integral converges and equals 2.

Is the integral $\int_{-\infty}^0 e^{x}\, dx$ convergent, and what is its value?

Yes, $\int_{-\infty}^0 e^{x}\, dx$ converges. Its value is $[e^{x}]_{-\infty}^0 = 1 - 0 = 1$.

How can I determine whether the improper integral $\int_1^\infty 1/(x^p)\, dx$ converges for different values of p?

For $p > 1$, the integral $\int_1^\infty 1/x^p\, dx$ converges and equals $1/(p-1)$. For $p \leq 1$, it diverges because the integral behaves like the harmonic or similar divergent integrals.

Additional Resources

Determine Whether The Following Improper Integral Converges Or Diverges. If It Converges, Find Its Value.

In the realm of calculus, improper integrals represent a fascinating class of integrals that extend the concept of definite integrals to unbounded domains or integrands with unbounded behavior. Understanding whether such integrals converge or diverge is fundamental, both theoretically and practically, as it underpins numerous applications across physics, engineering, and mathematical analysis. This article provides a comprehensive investigation into the methods for determining convergence or divergence of improper integrals, illustrating the principles through detailed examples, and demonstrating techniques to evaluate their values when they converge.

Introduction to Improper Integrals

An improper integral typically arises in two primary contexts:

1. Infinite limits of integration, such as integrals over $[a, \infty)$ or $(-\infty, b]$.
2. Unbounded integrands, where the integrand becomes infinite at some point within the interval of integration.

Mathematically, an improper integral may be written as:

- $\displaystyle \int_a^{\infty} f(x) \, dx$,
- $\displaystyle \int_{-\infty}^b f(x) \, dx$,
- $\displaystyle \int_a^b f(x) \, dx$, where $f(x)$ has an infinite discontinuity within $[a, b]$.

Determining whether these integrals converge involves examining their limits and behavior of the integrand near points of potential divergence.

Criteria for Convergence and Divergence

Convergence at Infinite Limits

For integrals over unbounded domains such as $\int_a^{\infty} f(x) dx$, the integral converges if the limit:

$$
\lim_{t \to \infty} \int_a^t f(x) \, dx
$$

exists and is finite. Similarly, for $\int_{-\infty}^b f(x) dx$, convergence depends on the limit as $t \to -\infty$.

Convergence at Finite Discontinuities

When the integrand has an infinite discontinuity at a point $c \in (a, b)$, the integral is split into two parts:

$$
\int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx,
$$

and convergence depends on the finiteness of these limits as the integrand approaches the discontinuity.

Comparison and Limit Comparison Tests

To analyze convergence, especially when the integrand behaves like a known function at infinity or near a discontinuity, the comparison tests are invaluable:

- Comparison Test: If $0 \leq f(x) \leq g(x)$ for sufficiently large x, and $\int_a^{\infty} g(x) dx$ converges, then $\int_a^{\infty} f(x) dx$ converges.
- Limit Comparison Test: If $\lim_{x \to \infty} \frac{f(x)}{g(x)} = c$, where c is a finite positive constant, then either both integrals $\int_a^{\infty} f(x) dx$ and $\int_a^{\infty} g(x) dx$ converge or both diverge.

Integral Tests for Specific Functions

Certain classes of functions, such as power functions $f(x) = \frac{1}{x^p}$, are well-understood, and their convergence depends on the exponent p:

- $\int_a^{\infty} \frac{1}{x^p} dx$ converges if and only if $p > 1$.
- $\int_0^{a} \frac{1}{x^p} dx$ converges if and only if $p < 1$.

Analyzing Sample Improper Integrals

To illustrate the comprehensive approach to these problems, consider the following representative integrals.

Example 1: $\displaystyle \int_1^{\infty} \frac{1}{x^2} dx$

Step 1: Recognize the form

This is a classic p-integral with $p=2$.

Step 2: Evaluate the integral

\[

$$
\int_1^{t} \frac{1}{x^2} dx = \left[-\frac{1}{x} \right]_1^{t} = -\frac{1}{t} + 1.
$$

Step 3: Take the limit as $t \to \infty$

$$
\lim_{t \to \infty} \left(-\frac{1}{t} + 1 \right) = 0 + 1 = 1.
$$

Conclusion: The integral converges, and its value is

$$
\boxed{\int_1^{\infty} \frac{1}{x^2} dx = 1}.
$$

Example 2: $\displaystyle \int_1^{\infty} \frac{1}{x} dx$

Step 1: Recognize the form

This is the harmonic integral with $p=1$.

Step 2: Evaluate the integral

$$
\int_1^{t} \frac{1}{x} dx = \left[\ln|x| \right]_1^{t} = \ln t - 0 = \ln t.
$$

Step 3: Limit as $t \to \infty$

$$
\lim_{t \to \infty} \ln t = \infty.
$$

Conclusion: The integral diverges.

Example 3: $\displaystyle \int_0^{1} \frac{1}{x^{1/2}} dx$

Step 1: Recognize the form

Near zero, the integrand behaves like $x^{-1/2}$.

Step 2: Evaluate the integral

$$
\int_0^{t} x^{-1/2} dx = \left[2 x^{1/2} \right]_0^{t} = 2 t^{1/2} - 0.
$$

Step 3: Take the limit as $t \to 1$

Finite; the integral is finite over $[0,1]$.

Conclusion: The integral converges, and its value over $[0,1]$ is

$$
\int_0^{1} \frac{1}{x^{1/2}} dx = 2.
$$

General Strategies for Evaluating Improper Integrals

1. Use of Comparison Tests

Identify dominant behavior of the integrand near points of potential divergence, and compare with known convergent or divergent integrals.

2. Substitution Methods

Substituting variables to transform the integral into a standard form can simplify the analysis, especially when dealing with powers or exponential functions.

3. Limit Techniques

Evaluate the limit of the integral as the upper or lower bounds approach infinity or a point of discontinuity.

4. Recognizing Standard Integral Forms

Many improper integrals reduce to known forms involving exponential, logarithmic, or power functions, allowing for straightforward evaluation.

When the Integral Converges: Finding Its Value

Once convergence is established, the next step involves computing the value of the integral. This often involves:

- Direct integration using antiderivatives.
- Applying limits carefully, especially at infinity or points of discontinuity.
- Using substitution to simplify the integrand.

For example, consider the integral:

$$
\int_0^{\infty} \frac{\sin x}{x} dx.
$$

This integral converges (conditionally) and has a well-known value:

$$
\boxed{\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}}.
$$

This showcases the importance of recognizing special integrals and employing advanced techniques like Fourier analysis or contour integration when necessary.

Practical Implications and Applications

Understanding whether improper integrals converge or diverge is crucial in various scientific disciplines:

- Physics: Calculating total energy, charge distributions, or probability densities often involves improper integrals.
- Engineering: Signal processing and control systems frequently involve integrals over infinite domains.
- Probability Theory: Expectations and variances of distributions may be expressed as improper integrals.

Accurate determination of convergence ensures the validity of models and calculations, preventing misinterpretations or errors.

Conclusion

Determining whether an improper integral converges or diverges is a fundamental skill in advanced calculus, supported by a suite of analytical tools and criteria. Recognizing the behavior of the integrand near points of potential divergence, employing comparison and limit comparison tests, and leveraging known integral forms are essential strategies.

When convergence is established, calculating the integral's value often involves straightforward antiderivatives or more sophisticated techniques, especially for complex functions. Awareness of convergence properties not only enriches one's mathematical understanding but also underpins applications across scientific disciplines, emphasizing the importance of mastery in analyzing improper integrals.

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