

Find A, B And C So That The Quadrature Formula Has The Highest Degree Of Precision $\int_a^b f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$

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In the realm of numerical analysis, quadrature formulas are essential tools for approximating definite integrals, especially when the antiderivative of the integrand is difficult or impossible to obtain analytically. The goal is to select specific points and weights—represented by parameters A, B, and C—that will maximize the degree of precision of the quadrature rule. This means designing a formula that exactly integrates polynomials of as high a degree as possible, ensuring accuracy and efficiency for a broad class of functions.

This article delves into the process of determining the optimal values of A, B, and C in a quadrature formula to achieve the highest degree of precision when approximating the integral of a function $f(x)$ over a given interval, typically $[0,1]$. We will explore the theoretical foundations, derive the necessary conditions, and present practical strategies for finding A, B, and C, ensuring the quadrature formula's optimal performance.

Understanding Quadrature Formulas and Degree of Precision

What is a Quadrature Formula?

A quadrature formula approximates the definite integral of a function $f(x)$ over an interval $[a, b]$ by a weighted sum of the function's values at specific points (nodes):

$$\int_a^b f(x) dx \approx \sum_{i=1}^n w_i f(x_i)$$

where:

- x_i are the nodes within $[a, b]$,
- w_i are the corresponding weights.

The choice of nodes and weights significantly influences the accuracy of the approximation.

Degree of Precision

The degree of precision (or degree of exactness) of a quadrature formula is the highest degree (d) such that the formula exactly integrates all polynomials $(f(x))$ of degree $(\leq d)$. For instance, a quadrature formula with degree of precision 3 will perfectly integrate all constant, linear, quadratic, and cubic polynomials.

Achieving the highest possible degree of precision is desirable because it guarantees the best accuracy for smooth functions that can be well-approximated by polynomials.

Formulating the Quadrature Rule with Parameters A, B, and C

Suppose we are considering a three-point quadrature formula over the interval $([0, 1])$:

$$\int_0^1 f(x) \, dx \approx A f(\alpha) + B f(\beta) + C f(\gamma)$$

where:

- (α, β, γ) are the nodes within $([0, 1])$,
- (A, B, C) are the weights associated with these nodes.

Often, the nodes are chosen symmetrically or based on specific rules (like Gaussian quadrature). However, in this context, we treat A, B, and C as parameters to be determined to maximize the degree of precision.

Deriving Conditions for Maximum Degree of Precision

Step 1: Ensuring Exactness for Polynomials of Low Degree

To maximize the degree of precision, impose the condition that the quadrature rule exactly integrates all polynomials up to degree (d) . This leads to a system of equations based on moments:

$$\int_0^1 x^k dx = \sum_{i=1}^3 w_i x_i^k, \quad k=0, 1, 2, \dots, d$$

For example:

- For $(k=0)$:

$$\int_0^1 1 dx = 1 = A + B + C$$

- For $(k=1)$:

$$\int_0^1 x dx = \frac{1}{2} = A \alpha + B \beta + C \gamma$$

- For $(k=2)$:

$$\int_0^1 x^2 dx = \frac{1}{3} = A \alpha^2 + B \beta^2 + C \gamma^2$$

Similarly, for higher degrees, the equations continue, but to determine A, B, and C, typically only the first few moments are used.

Step 2: Expressing the Conditions in Terms of A, B, and C

Assuming the nodes (α, β, γ) are fixed or chosen based on context, the problem reduces to solving:

$$\begin{cases} A + B + C = 1 \\ A \alpha + B \beta + C \gamma = \frac{1}{2} \\ A \alpha^2 + B \beta^2 + C \gamma^2 = \frac{1}{3} \end{cases}$$

The goal is to find A, B, and C such that these equations hold, and the resulting quadrature rule has the highest possible degree of precision.

Strategies to Maximize the Degree of Precision

1. Choosing Nodes to Simplify Calculations

Common choices for nodes include:

- Uniform nodes: $(\alpha=0, \beta=\frac{1}{2}, \gamma=1)$
- Gaussian nodes: Roots of Legendre polynomials
- Symmetric nodes: $(\alpha=0, \beta=\frac{1}{2}, \gamma=1)$

Selecting appropriate nodes simplifies the calculations and often leads to well-known quadrature rules like Simpson's rule or Gaussian quadrature.

2. Solving the System for A, B, and C

Once nodes are selected, solve the linear system:

$$\begin{cases} A + B + C = 1 \\ A\alpha + B\beta + C\gamma = \frac{1}{2} \\ A\alpha^2 + B\beta^2 + C\gamma^2 = \frac{1}{3} \end{cases}$$

to determine weights that ensure exact integration up to the quadratic polynomial.

3. Verifying the Degree of Precision

After solving for A, B, and C, verify whether the quadrature rule integrates higher-degree polynomials exactly by testing:

$$\int_0^1 x^k dx \stackrel{?}{=} \sum_{i=1}^3 w_i x_i^k$$

for $(k=3, 4, \dots)$. The highest (k) for which the equality holds indicates the degree of precision.

Practical Example: Simpson's Rule as a Special Case

Suppose we want to recover Simpson's rule within this framework. The nodes are:

$$\backslash[\\ \alpha=0, \quad \beta=\frac{1}{2}, \quad \gamma=1 \\ \backslash]$$

The weights are known:

$$\backslash[\\ A = \frac{1}{6}, \quad B = \frac{4}{6} = \frac{2}{3}, \quad C = \frac{1}{6} \\ \backslash]$$

Check the moments:

- $(k=0)$:

$$\backslash[\\ A + B + C = \frac{1}{6} + \frac{2}{3} + \frac{1}{6} = 1 \\ \backslash]$$

- $(k=1)$:

$$\backslash[\\ 0 \times \frac{1}{6} + \frac{1}{2} \times \frac{2}{3} + 1 \times \frac{1}{6} = 0 + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} \\ \backslash]$$

- $(k=2)$:

$$\backslash[\\ 0^2 \times \frac{1}{6} + \left(\frac{1}{2}\right)^2 \times \frac{2}{3} + 1^2 \times \frac{1}{6} = 0 + \frac{1}{4} \times \frac{2}{3} + \frac{1}{6} = 0 + \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \\ \backslash]$$

Simpson's rule exactly integrates polynomials up to degree 3, so the degree of precision here is 3.

Advanced Considerations: Gaussian Quadrature and Optimality

Gaussian quadrature provides the highest degree of precision for a given number of nodes, with nodes chosen as roots of orthogonal polynomials (e.g., Legendre polynomials for $[0,1]$). For three nodes, Gaussian quadrature achieves degree 5 precision.

To match this level of optimality with parameters A, B, and C, one would:

- Choose nodes as roots of the Legendre polynomial $(P_3(x))$,
- Calculate weights via known formulas involving orthogonal polynomials,
- Ensure the resulting formula integrates polynomials up to degree 5 exactly.

This process involves more advanced techniques, such as solving for roots of Legendre polynomials and applying quadrature formulas derived from orthogonality properties.

Conclusion: How to Find A, B, and C for

Frequently Asked Questions

What is the goal when finding A, B, and C in the quadrature formula for $\int_0^1 F(x) dx$?

The goal is to determine values of A, B, and C such that the quadrature formula has the highest possible degree of precision, meaning it exactly integrates polynomials up to the highest degree.

How do we define the degree of precision in a quadrature formula?

The degree of precision is the highest degree n for which the quadrature formula exactly integrates all polynomials of degree $\leq n$.

What conditions must A, B, and C satisfy to maximize the degree of precision?

They must satisfy equations derived from matching the quadrature formula's moments with those of the integral for polynomials of increasing degree, typically starting with constant, linear, and quadratic functions.

How can we set up equations to find A, B, and C for the quadrature formula?

By requiring the formula to be exact for monomials x^0 , x^1 , and x^2 , leading to a system of equations that A, B, and C must satisfy.

What is the process for determining the highest degree of precision for a given quadrature formula?

Test the formula against polynomials of increasing degree, ensuring it is exact for each, until it fails, thus identifying the maximum degree of precision.

Can the quadrature formula be further refined to increase its degree of precision?

Yes, by choosing additional nodes or adjusting coefficients, but in this case, fixing A, B, and C constrains the degree of precision based on the current setup.

Are symmetry considerations relevant when determining A, B, and C?

Yes, for symmetric intervals like $[0,1]$, symmetry can simplify calculations and sometimes lead to more accurate or higher-precision formulas.

What role does the choice of nodes play in maximizing the degree of precision?

Selecting optimal nodes (points where function values are sampled) can enhance the degree of precision; in some cases, fixing nodes simplifies the determination of coefficients.

What is the typical approach to solve for A, B, and C in practice?

Set up a system of equations from the conditions for exactness on low-degree polynomials and solve for A, B, and C using algebraic or numerical methods.

Why is it important to maximize the degree of precision in quadrature formulas?

A higher degree of precision ensures more accurate integration of a wider class of functions with fewer evaluation points, improving efficiency and reliability of numerical integration.

Additional Resources

Find A, B And C So That The Quadrature Formula Has The Highest Degree Of Precision Integral $\int_a^b f(x) dx \approx \frac{b-a}{3} [A f(a) + B f(\frac{a+b}{2}) + C f(b)]$

Numerical integration, also known as quadrature, is a fundamental technique in applied mathematics and computational science that allows us to approximate definite integrals when an analytical solution is difficult or impossible to obtain. When constructing quadrature formulas, one of the key objectives is to maximize the degree of precision — that is, to ensure the formula exactly integrates polynomials up to as high a degree as possible. The problem of determining the weights and nodes for a quadrature rule, specifically the weights A, B, and C in this context, so that the formula achieves the

highest degree of precision for the integral of a function $(F(x))$ over $([0, 1])$, is both classical and rich in mathematical insight.

Understanding Quadrature Formulas and Degree of Precision

What is a Quadrature Formula?

A quadrature formula approximates the definite integral of a function $(F(x))$ over an interval $([a, b])$ in the form:

$$\int_a^b F(x) \, dx \approx \sum_{i=1}^n w_i F(x_i)$$

where:

- (x_i) are the nodes or points at which the function is evaluated.**
- (w_i) are the weights associated with each node.**

The goal is to choose the nodes and weights so

that the approximation is as accurate as possible for a broad class of functions.

Degree of Precision

The degree of precision (or exactness) of a quadrature formula is the highest degree (d) such that the formula integrates all polynomials $(p(x))$ of degree less than or equal to (d) exactly:

$$\int_a^b p(x) \, dx = \sum_{i=1}^n w_i p(x_i),$$

quad {for all } p(x) { with } \deg(p) \leq d

Maximizing this degree is a central goal because it guarantees accurate integration of polynomial functions up to that degree and provides insight into how well the formula approximates more general functions.

Formulating the Problem: Finding A, B, and C

Suppose we are considering a simple quadrature

rule with three nodes: (x_1, x_2, x_3) within $[0, 1]$. The weights are (A, B, C) , respectively. The quadrature formula can be written as:

$$\int_0^1 F(x) \, dx \approx A \cdot F(x_1) + B \cdot F(x_2) + C \cdot F(x_3)$$

Our task is to determine the nodes (x_1, x_2, x_3) and weights (A, B, C) such that the degree of precision of this rule is maximized.

Constructing the System of Equations

Matching Moments for Exactness

To ensure the quadrature formula is exact for polynomials up to degree (d) , we impose the following conditions:

$$\begin{cases} \int_0^1 1 \, dx = A + B + C \\ \int_0^1 x \, dx = A x_1 + B x_2 + C x_3 \end{cases}$$

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\int_0^1 x^2 \, dx = A x_1^2 + B x_2^2 + C x_3^2
\\
\int_0^1 x^3 \, dx = A x_1^3 + B x_2^3 + C x_3^3
\\
\vdots
\end{cases}
\]

```

Since the integral over $([0,1])$ of (x^k) is $(\frac{1}{k+1})$, the conditions become:

```

\|
\begin{cases}
A + B + C = 1 \\
A x_1 + B x_2 + C x_3 = \frac{1}{2} \\
A x_1^2 + B x_2^2 + C x_3^2 = \frac{1}{3} \\
A x_1^3 + B x_2^3 + C x_3^3 = \frac{1}{4}
\end{cases}
\|

```

Our goal is to choose (x_1, x_2, x_3) and corresponding weights (A, B, C) satisfying these equations to maximize the degree (d) .

Optimizing the Nodes and Weights for Highest Degree of Precision

Classical Approaches

The problem is akin to constructing Gaussian quadrature rules, which are known to maximize the degree of precision for a given number of nodes. For three nodes, the optimal choice (under certain symmetry assumptions) is often the Gauss-Legendre nodes, which are roots of Legendre polynomials scaled to $[0, 1]$.

However, in this case, because the interval is $[0, 1]$, and the nodes are not necessarily symmetric, one approach is to impose symmetry or to solve the system directly for general nodes.

Choosing Nodes for Maximum Precision

- Symmetric Nodes: Setting $x_1 = 0$, $x_2 = 0.5$, and $x_3 = 1$ simplifies calculations but may limit the degree of precision.**
- Optimal Nodes: To maximize the degree, the nodes should be the roots of orthogonal polynomials (e.g., Legendre polynomials) mapped onto $[0, 1]$.**

By solving the system for the weights once the nodes are fixed, we can verify the degree of precision.

Explicit Solution for A, B, and C

Suppose we select nodes (x_1, x_2, x_3) . The weights are determined by solving:

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$$

This system can be solved explicitly using Cramer's rule or matrix inversion to find the weights:

$\begin{bmatrix}$

```

A = \frac{
\det
\begin{bmatrix}
1 & 1 & 1 \\\
\frac{1}{2} & x_2 & x_3 \\\
\frac{1}{3} & x_2^2 & x_3^2
\end{bmatrix}
\{
\det
\begin{bmatrix}
1 & 1 & 1 \\\
x_1 & x_2 & x_3 \\\
x_1^2 & x_2^2 & x_3^2
\end{bmatrix}\}
\}

```

and similarly for (B) and (C) . By choosing the nodes to be the roots of the Legendre polynomial (P_3) scaled to $([0,1])$, the corresponding weights will give the highest degree of precision, which for three nodes is 5.

Features, Pros, and Cons of Various Quadrature Choices

Features:

- **Gaussian Quadrature:** Nodes are roots of orthogonal polynomials; maximizes degree of precision.
- **Newton-Cotes Rules:** Equally spaced nodes; easier to implement but lower degree of precision for larger n .
- **Clenshaw-Curtis Quadrature:** Uses Chebyshev nodes; good for integrand oscillations.

Pros of Using Gaussian Quadrature:

- Achieves the highest possible degree of precision for a given number of nodes.
- Efficient for smooth functions.

Cons:

- Nodes are not equally spaced, making implementation slightly complex.
- Computing nodes involves solving polynomial equations.
- Less robust for functions with singularities or discontinuities.

Practical Implementation and Numerical Considerations

In practice, determining (A, B, C) involves:

- Computing the nodes, often via root-finding algorithms for orthogonal polynomials.**
- Solving the linear system to find weights.**
- Validating the degree of precision by testing the formula on polynomials of degrees up to (d) .**

For instance, the classical three-point Gaussian quadrature on $([0,1])$ involves transforming the roots of the Legendre polynomial (P_3) from $([-1,1])$ to $([0,1])$, then computing weights accordingly.

Conclusion

Finding (A, B, C) so that the quadrature formula has the highest degree of precision is a nuanced task that combines polynomial theory, numerical linear algebra, and approximation theory. The key steps involve selecting optimal nodes—often roots of orthogonal polynomials—then solving the associated moment equations to determine the weights. Gaussian quadrature rules exemplify this

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