Find The Arc Length Of Y=((x+2)/2)^4+1/(2(x+2)^2) Over [1,4].(Give An Exact Answer. Use Symbolic Notation

Find The Arc Length Of Y=((x+2)/2)^4+1/(2(x+2)^2) Over [1,4].(Give An Exact Answer. Use Symbolic Notation

Calculating the arc length of a curve is a fundamental problem in calculus that involves integrating the length of infinitesimal tangent segments along the curve. In this article, we will focus on determining the exact arc length of the function

$$ Y = \left(\frac{x+2}{2}\right)^4 + \frac{1}{2(x+2)^2} $$

over the interval $[1, 4]$. Our goal is to provide a precise, symbolic solution for this problem, emphasizing clarity and mathematical rigor.

Understanding Arc Length and Its Formula

The Concept of Arc Length

Arc length measures the distance along a curve between two points. For a smooth function $y = f(x)$, the arc length L from $x = a$ to $x = b$ is given by the integral:

$$ L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx $$

This formula stems from the Pythagorean theorem, where $\sqrt{1 + (dy/dx)^2}$ represents the infinitesimal element of arc length ds.

Applying the Formula to the Given Function

To compute the arc length of $Y = f(x)$, we first need to find $\frac{dy}{dx}$, then evaluate the integral:

$$ L = \int_1^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx $$

The challenge lies in simplifying $\left(\frac{dy}{dx}\right)^2$ and recognizing potential algebraic patterns or identities to evaluate the integral exactly.

Step-by-Step Solution for the Given Function

1. Express the function explicitly

Given:

$$ y = \left(\frac{x+2}{2}\right)^4 + \frac{1}{2(x+2)^2} $$

For clarity, rewrite as:

$$ y = \left(\frac{1}{2}\right)^4 (x+2)^4 + \frac{1}{2} (x+2)^{-2} $$

which simplifies to:

$$ y = \frac{(x+2)^4}{16} + \frac{1}{2(x+2)^2} $$

2. Compute the derivative dy/dx

Using the power rule:

$$
\frac{dy}{dx} = \frac{d}{dx} \left(\frac{(x+2)^4}{16} \right) + \frac{d}{dx} \left(\frac{1}{2} (x+2)^{-2} \right)
$$

Calculate each term separately:

- First term:

$$
\frac{d}{dx} \left(\frac{(x+2)^4}{16} \right) = \frac{1}{16} \cdot 4 (x+2)^3 = \frac{(x+2)^3}{4}
$$

- Second term:

$$
\frac{d}{dx} \left(\frac{1}{2} (x+2)^{-2} \right) = \frac{1}{2} \cdot (-2) (x+2)^{-3} = -(x+2)^{-3}
$$

Combine:

$$
\frac{dy}{dx} = \frac{(x+2)^3}{4} - \frac{1}{(x+2)^3}
$$

3. Compute $\left(\frac{dy}{dx}\right)^2$

Set:

$$
u = \frac{(x+2)^3}{4} \quad \text{and} \quad v = \frac{1}{(x+2)^3}
$$

then

$$
\left(\frac{dy}{dx}\right)^2 = (u - v)^2 = u^2 - 2uv + v^2
$$

Calculate each:

- $u^2 = \left(\frac{(x+2)^3}{4}\right)^2 = \frac{(x+2)^6}{16}$

- $v^2 = \left(\frac{1}{(x+2)^3}\right)^2 = \frac{1}{(x+2)^6}$

- $2uv = 2 \times \frac{(x+2)^3}{4} \times \frac{1}{(x+2)^3} = 2 \times \frac{(x+2)^3}{4} \times \frac{1}{(x+2)^3} = 2 \times \frac{1}{4} = \frac{1}{2}$

Putting these together:

$$
\left(\frac{dy}{dx}\right)^2 = \frac{(x+2)^6}{16} - \frac{1}{2} + \frac{1}{(x+2)^6}
$$

4. Simplify $1 + \left(\frac{dy}{dx}\right)^2$

Add 1:

$$
1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{(x+2)^6}{16} - \frac{1}{2} + \frac{1}{(x+2)^6}
$$

Express 1 as $\frac{16}{16}$:

$$
= \frac{16}{16} + \frac{(x+2)^6}{16} - \frac{8}{16} + \frac{1}{(x+2)^6}
$$

Combine:

$$
= \frac{(x+2)^6 + 16 - 8}{16} + \frac{1}{(x+2)^6} = \frac{(x+2)^6 + 8}{16} + \frac{1}{(x+2)^6}

\]

Now, write \(\frac{1}{(x+2)^6} \) with denominator 16:

\[
\frac{1}{(x+2)^6} = \frac{16}{16 (x+2)^6}
\]

So,

\[
1 + \left(\frac{dy}{dx}\right)^2 = \frac{(x+2)^6 + 8}{16} + \frac{16}{16 (x+2)^6}
\]

Express both as a common denominator:

\[
= \frac{(x+2)^6 + 8}{16} + \frac{16}{16 (x+2)^6}
\]

Note that:

\[
\frac{16}{16 (x+2)^6} = \frac{1}{(x+2)^6}
\]

which we already have; thus, the entire expression is:

\[
= \frac{(x+2)^6 + 8 + 16}{16} = \frac{(x+2)^6 + 24}{16}
\]

Therefore,

\[
1 + \left(\frac{dy}{dx}\right)^2 = \frac{(x+2)^6 + 24}{16}
\]

5. Compute the integrand \(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \)

\[
\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{\frac{(x+2)^6 + 24}{16}} = \frac{\sqrt{(x+2)^6 + 24}}{4}
\]

Evaluating the Arc Length Integral

The total arc length L over $[1, 4]$ is:

$$
L = \int_1^4 \frac{\sqrt{(x+2)^6 + 24}}{4} \, dx
$$

Pull out the constant:

$$
L = \frac{1}{4} \int_1^4 \sqrt{(x+2)^6 + 24} \, dx
$$

Substitution:

Let:

$$
t = x + 2 \Rightarrow dt = dx
$$

When $x = 1$, $t = 3$; when $x = 4$, $t = 6$. The integral becomes:

$$
L = \frac{1}{4} \int_3^6 \sqrt{t^6 + 24} \, dt
$$

Expressing the integral:

$$
L = \frac{1}{4} \int_3^6 \sqrt{t^6 + 24} \, dt
$$

This integral involves $\sqrt{t^6 + 24}$, which suggests using substitution or recognizing a pattern for exact evaluation.

Recognizing the structure

Note that:

$$
t^6 + 24 =
$$

Frequently Asked Questions

How do you set up the arc length integral for the function y = ((x+2)/2)^4 + 1/(2(x+2)^2) over [1,4]?

The arc length L is given by L = ∫[1 to 4] √[1 + (dy/dx)^2] dx. First, compute dy/dx, then substitute into the integral to find the exact length.

What is the derivative dy/dx of y = ((x+2)/2)^4 + 1/(2(x+2)^2)?

dy/dx = (x+2)^3 / 2 + (x+2)^{-3}. This is obtained by differentiating each term separately using the chain rule and power rule.

How do you simplify the expression under the square root for the arc length calculation?

Compute (dy/dx)^2 = ((x+2)^3 / 2 + (x+2)^{-3})^2, then combine like terms to simplify the expression before integrating.

What substitutions can be used to evaluate the integral for the arc length exactly?

Let t = x + 2, transforming the limits to t=3 and t=6, and rewrite the integral in terms of t to simplify the algebra and facilitate exact symbolic integration.

What is the exact symbolic form of the arc length integral after substitution?

L = ∫[3 to 6] √[1 + (t^3 / 2 + t^{-3})^2] dt, which can be expanded and simplified further to evaluate symbolically.

What is the final exact expression for the arc length of y = ((x+2)/2)^4 + 1/(2(x+2)^2) over [1,4]?

L = ∫[3 to 6] √[1 + (t^3 / 2 + t^{-3})^2] dt, which can be expressed in closed form using symbolic integration techniques, but typically involves complex algebraic expressions involving radicals and rational functions.

Additional Resources

Arc Length Calculation of the Function \(y = \left(\frac{x+2}{2}\right)^4 + \frac{1}{2(x+2)^2} \) over the interval \([1, 4]\)

Introduction

Calculating the arc length of a function over a specified interval is a fundamental problem in calculus, often encountered in physics, engineering, and mathematical analysis. The precise measurement of a curve's length provides insights into the geometry of the curve and its physical properties when representing real-world phenomena, such as the length of a wire, the trajectory of an object, or the surface area of a shape.

In this article, we examine the function:

$$
y = \left(\frac{x+2}{2}\right)^4 + \frac{1}{2(x+2)^2}
$$

and aim to find its exact arc length over the interval $[1, 4]$. Our approach will involve symbolic calculus techniques, emphasizing the derivation of the exact integral and its evaluation.

Understanding the Problem

The Function and Its Domain

Given the function:

$$
y(x) = \left(\frac{x+2}{2}\right)^4 + \frac{1}{2(x+2)^2}
$$

- Domain: $x \in [1, 4]$
- The function combines polynomial and rational components, which suggests potential for simplification through algebraic manipulation.

Goal

Compute the exact arc length L of $y(x)$ over $[1, 4]$, given by:

$$
L = \int_{1}^{4} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

Step 1: Simplify the Function

Rewriting the Function

To facilitate differentiation, express y in a clearer form:

\[
y(x) = \left(\frac{x+2}{2} \right)^4 + \frac{1}{2(x+2)^2}
\]

Note that:

\[
\left(\frac{x+2}{2} \right)^4 = \frac{(x+2)^4}{16}
\]

Thus:

\[
y(x) = \frac{(x+2)^4}{16} + \frac{1}{2(x+2)^2}
\]

Step 2: Derive \(\frac{dy}{dx} \)

Differentiation

We differentiate each term separately using the chain rule:

\[
\frac{dy}{dx} = \frac{d}{dx} \left(\frac{(x+2)^4}{16} \right) + \frac{d}{dx} \left(
\frac{1}{2(x+2)^2} \right)
\]

- For the first term:

\[
\frac{d}{dx} \left(\frac{(x+2)^4}{16} \right) = \frac{4 (x+2)^3}{16} = \frac{(x+2)^3}{4}
\]

- For the second term:

\[
\frac{d}{dx} \left(\frac{1}{2(x+2)^2} \right)
\]

Recall that:

\[
\frac{d}{dx} \left((x+2)^{-2} \right) = -2 (x+2)^{-3}
\]

Therefore:

\[
\frac{d}{dx} \left(\frac{1}{2} (x+2)^{-2} \right) = \frac{1}{2} \times -2 (x+2)^{-3} = -

(x+2)^{-3}
\]

Final derivative:

\[
\boxed{
\frac{dy}{dx} = \frac{(x+2)^3}{4} - \frac{1}{(x+2)^3}
}
\]

Step 3: Compute \(1 + \left(\frac{dy}{dx} \right)^2 \)

The integrand involves:

\[
\sqrt{1 + \left(\frac{dy}{dx} \right)^2}
\]

Calculating \(\left(\frac{dy}{dx} \right)^2 \):

\[
\left(\frac{(x+2)^3}{4} - \frac{1}{(x+2)^3} \right)^2
\]

Let \(t = x + 2 \) to simplify notation; the interval \(x \in [1,4] \) becomes \(t \in [3, 6] \).

Expressed in \(t \):

\[
\frac{dy}{dx} = \frac{t^3}{4} - \frac{1}{t^3}
\]

Now, compute the square:

\[
\left(\frac{t^3}{4} - \frac{1}{t^3} \right)^2 = \left(\frac{t^3}{4} \right)^2 - 2 \times
\frac{t^3}{4} \times \frac{1}{t^3} + \left(\frac{1}{t^3} \right)^2
\]

Simplify each term:

- \(\left(\frac{t^3}{4} \right)^2 = \frac{t^6}{16} \)
- \(2 \times \frac{t^3}{4} \times \frac{1}{t^3} = 2 \times \frac{1}{4} \times 1 = \frac{1}{2} \)
- \(\left(\frac{1}{t^3} \right)^2 = \frac{1}{t^6} \)

Therefore:

\[

\left(\frac{dy}{dx} \right)^2 = \frac{t^6}{16} - \frac{1}{2} + \frac{1}{t^6}
\]

Now, add 1:

\[
1 + \left(\frac{dy}{dx} \right)^2 = 1 + \frac{t^6}{16} - \frac{1}{2} + \frac{1}{t^6}
\]

Simplify:

\[
= \left(1 - \frac{1}{2} \right) + \frac{t^6}{16} + \frac{1}{t^6} = \frac{1}{2} + \frac{t^6}{16}
+ \frac{1}{t^6}
\]

Step 4: Simplify the Expression Under the Square Root

We now consider:

\[
\sqrt{1 + \left(\frac{dy}{dx} \right)^2 } = \sqrt{ \frac{1}{2} + \frac{t^6}{16} + \frac{1}{t^6} }
\]

Our goal is to simplify this expression to an integrable form.

Combine terms over common denominators

Express all terms with denominator 16:

\[
\frac{1}{2} = \frac{8}{16}
\]

and

\[
\frac{1}{t^6} = \frac{16}{16 t^6}
\]

Thus:

\[
\sqrt{ \frac{8}{16} + \frac{t^6}{16} + \frac{16}{16 t^6} } = \sqrt{ \frac{8 + t^6 + 16 / t^6
}{16} }
\]

Take the square root:

$$
= \frac{1}{4} \sqrt{ 8 + t^6 + \frac{16}{t^6} }
$$

Focus on numerator:

$$
8 + t^6 + \frac{16}{t^6}
$$

Note the symmetry between t^6 and $16 / t^6$. To combine these, consider the expression:

$$
t^6 + \frac{16}{t^6}
$$

which resembles the sum of a quantity and its reciprocal scaled appropriately.

Step 5: Recognize a Perfect Square

Observe that:

$$
t^6 + \frac{16}{t^6} = \left(t^3 \right)^2 + 4^2 \times \frac{1}{t^6}
$$

Alternatively, note that:

$$
t^6 + \frac{16}{t^6} = \left(t^3 - \frac{4}{t^3} \right)^2 + 2 \times t^3 \times \frac{4}{t^3}
$$

But more straightforwardly, consider the identity:

$$
\left(t^3 - \frac{4}{t^3} \right)^2 = t^6 - 2 \times 4 + \frac{16}{t^6}
$$

which simplifies to:

$$
t^6 - 8 + \frac{16}{t^6}
$$

Adding 8:

$$
t^6 + \frac{16}{t^6} = \left(t^3 - \frac{4}{t^3} \right)^2 + 8
$$

Find The Arc Length Of Y X 2 2 4 1 2 X 2 2 Over 1 4 Give An Exact Answer Use Symbolic Notation

Find The Arc Length Of Y X 2 2 4 1 2 X 2 2 Over 1 4 Give An Exact Answer Use Symbolic Notation

Related Articles

- Describe How You Would Format A Cell So That If Its Value Is Less Than 20 The Cell Would Be Automatically
- At A Store, Socks Are Sold With A Different Number Of Socks In Each Package. Which Package Has The Lowest
- For Which Type Of Environment Should Organizations Adopt An Organic Structure?A) Munificent EnvironmentB)

Back to Home