Find The Area Of The Parallelogram With Sides A = 7i+2j+ K And B = 4i + 3j + K. (Express Numbers In Exact

Find The Area Of The Parallelogram With Sides A = 7i+2j+ K And B = 4i + 3j + K. (Express Numbers In Exact)

Understanding how to find the area of a parallelogram in three-dimensional space is a fundamental concept in vector calculus and physics. When given two vectors representing the sides of a parallelogram, the area can be computed using the cross product of these vectors. This article provides a comprehensive guide to calculating the area of a parallelogram with sides A and B, given in vector form, and emphasizes expressing all numbers in their exact form.

Introduction to Vectors and Parallelograms

Before diving into the calculation, it's essential to understand the foundational concepts:

- Vectors: Quantities having both magnitude and direction, represented in component form.
- Parallelogram in Space: A four-sided figure formed by two vectors originating from the same point, with opposite sides parallel and equal in length.

Given vectors:

- $\mathbf{A} = 7\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
- $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

Our goal is to find the area of the parallelogram spanned by these vectors.

Mathematical Approach to Find the Area

The area of a parallelogram formed by two vectors $\mathbf{A}$ and $\mathbf{B}$ in three-dimensional space is equal to the magnitude (length) of their cross product:

$$
\text{Area} = |\mathbf{A} \times \mathbf{B}|
$$

Key steps:

1. Compute the cross product $\mathbf{A} \times \mathbf{B}$.
2. Find the magnitude of the resulting vector.
3. Express the result in exact form.

Calculating the Cross Product

Given vectors:

$$
\mathbf{A} = (7, 2, 1), \quad \mathbf{B} = (4, 3, 1)
$$

The cross product formula:

$$
\mathbf{A} \times \mathbf{B} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
A_x & A_y & A_z \\
B_x & B_y & B_z
\end{vmatrix}
$$

Calculating determinants:

$$
\mathbf{A} \times \mathbf{B} = \mathbf{i} (A_y B_z - A_z B_y) - \mathbf{j} (A_x B_z - A_z B_x) + \mathbf{k} (A_x B_y - A_y B_x)
$$

Substitute the component values:

$$
\mathbf{A} \times \mathbf{B} = \mathbf{i} (2 \times 1 - 1 \times 3) - \mathbf{j} (7 \times 1 - 1 \times 4) + \mathbf{k} (7 \times 3 - 2 \times 4)
$$

Simplify each component:

- For $\mathbf{i}$:

$$
2 \times 1 - 1 \times 3 = 2 - 3 = -1
$$

- For $\mathbf{j}$:

$$
7 \times 1 - 1 \times 4 = 7 - 4 = 3
$$

- For $\mathbf{k}$:

$$
7 \times 3 - 2 \times 4 = 21 - 8 = 13
$$

Therefore,

$$
\mathbf{A} \times \mathbf{B} = -1 \mathbf{i} - 3 \mathbf{j} + 13 \mathbf{k}
$$

Expressed as a vector:

$$
\boxed{
\mathbf{A} \times \mathbf{B} = -\mathbf{i} - 3\mathbf{j} + 13\mathbf{k}
}
$$

Calculating the Magnitude of the Cross Product

The magnitude of the vector $\mathbf{A} \times \mathbf{B}$ is:

$$
|\mathbf{A} \times \mathbf{B}| = \sqrt{(-1)^2 + (-3)^2 + (13)^2}
$$

Compute each term:

- \((-1)^2 = 1 \)
- \((-3)^2 = 9 \)
- \((13)^2 = 169 \)

Sum:

\[
1 + 9 + 169 = 179
\]

Expressed in exact form, the magnitude is:

\[
\boxed{
|\mathbf{A} \times \mathbf{B}| = \sqrt{179}
\]

Thus, the area of the parallelogram is:

\[
\boxed{
\text{Area} = \sqrt{179}
}
\]

This is the exact answer, expressed in radical form.

Interpreting the Result

- The area is \(\sqrt{179} \) square units.
- Since 179 is a prime number, the radical cannot be simplified further.
- The simplicity of the radical underscores the importance of expressing numbers in exact form, especially in mathematical contexts where approximate decimal values might be misleading or undesirable.

Additional Insights and Applications

Understanding how to compute areas using vectors has multiple applications:

- Physics: Calculating the area of force vectors to determine torque.

- Computer Graphics: Computing surface areas in 3D models.
- Engineering: Analyzing the stability and forces in structures.
- Mathematics: Solving problems involving vector calculus and coordinate geometry.

Summary of the Calculation Process

1. Identify the vectors representing the sides of the parallelogram.
2. Compute the cross product using the determinant method.
3. Simplify the components to obtain the resulting vector.
4. Calculate the magnitude of the cross product vector.
5. Express the area in exact radical form.

Conclusion

By following the systematic approach of vector cross products and magnitude calculations, you can effectively determine the area of a parallelogram in 3D space. In the specific case where vectors are given as \(\mathbf{A} = 7\mathbf{i} + 2\mathbf{j} + \mathbf{k} \) and \(\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k} \), the area is precisely \(\sqrt{179} \) square units. Mastery of these vector operations is essential for advanced studies in mathematics, physics, and engineering, providing a foundation for solving more complex spatial problems.

Keywords: Vector cross product, Parallelogram area, 3D vectors, Vector calculus, Exact radical form, Space geometry

Frequently Asked Questions

How do you find the area of a parallelogram given two vectors?

The area of a parallelogram formed by vectors A and B is given by the magnitude of their cross product: Area = |A × B|.

What are the components of vectors A and B in the given problem?

Vector A = 7i + 2j + k, and Vector B = 4i + 3j + k.

How do you compute the cross product of vectors A and B in component form?

The cross product A × B is calculated using the determinant: |i j k|
|7 2 1|
|4 3 1|.

What is the step-by-step process to find the cross product of vectors A and B?

Calculate the determinant of the matrix with unit vectors i, j, k in the first row, components of A in the second, and components of B in the third. Then, compute each component:
$A \times B = ((2)(1) - (1)(3))i - ((7)(1) - (1)(4))j + ((7)(3) - (2)(4))k$.

What are the exact components of the cross product A × B?

$A \times B = (21 - 13)i - (71 - 14)j + (73 - 24)k = (2 - 3)i - (7 - 4)j + (21 - 8)k = (-1)i - (3)j + (13)k$.

How do you find the magnitude of the cross product to determine the area?

The magnitude is calculated as |A × B| = √((-1)^2 + (-3)^2 + 13^2) = √(1 + 9 + 169) = √179.

What is the exact area of the parallelogram in radical form?

The exact area is √179 square units.

How can the area be expressed in decimal form?

The approximate decimal value of √179 is about 13.38 units.

Why is it important to express the numbers in exact form in this problem?

Expressing in exact form preserves mathematical precision and allows for accurate further calculations or simplifications without rounding errors.

Additional Resources

Find The Area Of The Parallelogram With Sides A = 7i + 2j + k and B = 4i + 3j + k: An In-Depth

Analytical Perspective

The problem of determining the area of a parallelogram defined by two vectors in three-dimensional space is a fundamental exercise in vector calculus, with broad applications across physics, engineering, and mathematics. At its core, this task involves understanding the geometric interpretation of vectors, the calculation of their cross product, and the subsequent determination of the magnitude of this cross product to find the area. In this article, we delve into a comprehensive analysis of the problem, breaking down each step with meticulous detail, and presenting an exact, numerical solution to the given vectors.

Introduction to Vectors and Parallelogram Area

The Geometric Significance of Vectors A and B

In three-dimensional Euclidean space, vectors are entities characterized by both magnitude and direction. They are often represented in component form as ordered triples, such as:

- Vector A = 7i + 2j + k
- Vector B = 4i + 3j + k

Here, the components correspond to the coefficients of the unit vectors i, j, and k, which point along the x, y, and z axes, respectively. These vectors can be visualized as directed line segments originating from the origin to the points (7, 2, 1) and (4, 3, 1).

When considering the parallelogram formed by vectors A and B, the area is directly related to the magnitude of their cross product. The cross product yields a vector perpendicular to both A and B, with a magnitude equal to the area of the parallelogram spanned by these vectors.

Mathematical Foundations of the Area Calculation

The core mathematical principle underpinning this problem is:

$$\text{Area} = |\mathbf{A} \times \mathbf{B}|$$

where $\times$ denotes the cross product, and $|\cdot|$ indicates the magnitude of the resulting vector. The cross product of two vectors $\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k}$ and $\mathbf{B} = B_x \mathbf{i} + B_y \mathbf{j} + B_z \mathbf{k}$ is given by the determinant:

$$
\mathbf{A} \times \mathbf{B} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
A_x & A_y & A_z \\
B_x & B_y & B_z \\
\end{vmatrix}
$$

This determinant expands into a vector with components:

$$
\mathbf{A} \times \mathbf{B} = \left(A_y B_z - A_z B_y \right) \mathbf{i} - \left(A_x B_z - A_z B_x \right) \mathbf{j} + \left(A_x B_y - A_y B_x \right) \mathbf{k}
$$

The magnitude of this cross product vector is then calculated as:

$$
|\mathbf{A} \times \mathbf{B}| = \sqrt{ (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_x)^2 }
$$

This magnitude directly provides the exact area of the parallelogram.

Step-by-Step Calculation of the Cross Product

Component Breakdown

Given:

$$
\mathbf{A} = 7i + 2j + k \Rightarrow (A_x, A_y, A_z) = (7, 2, 1)
$$
$$
\mathbf{B} = 4i + 3j + k \Rightarrow (B_x, B_y, B_z) = (4, 3, 1)
$$

The components facilitate straightforward calculation of each part of the cross product.

Calculating the Components of $\mathbf{A} \times \mathbf{B}$

Using the determinant expansion:

- i-component:

$$
A_y B_z - A_z B_y = 2 \times 1 - 1 \times 3 = 2 - 3 = -1
$$

- j-component:

$$
A_x B_z - A_z B_x = 7 \times 1 - 1 \times 4 = 7 - 4 = 3
$$

- k-component:

$$
A_x B_y - A_y B_x = 7 \times 3 - 2 \times 4 = 21 - 8 = 13
$$

Putting it all together, the cross product vector is:

$$
\mathbf{A} \times \mathbf{B} = (-1) \mathbf{i} - (3) \mathbf{j} + (13) \mathbf{k}
$$

which simplifies to:

$$
\boxed{
\mathbf{A} \times \mathbf{B} = -1 \mathbf{i} - 3 \mathbf{j} + 13 \mathbf{k}
}
$$

Calculating the Magnitude of the Cross Product

Exact Numerical Computation

The magnitude of the cross product vector provides the precise area:

$$
|\mathbf{A} \times \mathbf{B}| = \sqrt{ (-1)^2 + (-3)^2 + 13^2 } = \sqrt{ 1 + 9 + 169 } = \sqrt{ 179 }
$$

Expressed in exact form, this is:

$$
\boxed{
\text{Area} = \sqrt{179}
$$

This is the most precise expression for the area, as $\sqrt{179}$ cannot be simplified further since 179 is a prime number.

Approximate Numerical Value

For practical applications, an approximate decimal value is often useful:

$$
\sqrt{179} \approx 13.379
$$

Hence, the area of the parallelogram is approximately 13.379 square units.

Analytical Significance and Applications

Implications of the Exact Area

Knowing the exact value of the area in radical form has theoretical importance. It ensures precise calculations in contexts where rounding errors can lead to inaccuracies, such as in engineering design, physics simulations, or computer graphics.

Furthermore, representing the area as $\sqrt{179}$ emphasizes the geometric relationship between the vectors, highlighting the inherent algebraic structure of the problem. This form respects the principles of mathematical elegance and accuracy.

Real-World Applications

- Physics: Calculating the area of a parallelogram formed by force vectors can determine work done or energy transfer in mechanical systems.
- Engineering: Structural analysis often involves computing areas and volumes based on vector components, especially in the design of trusses and frameworks.
- Computer Graphics: Determining the area of polygons in 3D space is crucial for rendering, shading, and collision detection.
- Mathematics Education: Problems like these reinforce understanding of vector operations, cross products, and their geometric interpretations.

Conclusion: Summarizing the Findings

The meticulous analysis of the vectors $\mathbf{A} = 7i + 2j + k$ and $\mathbf{B} = 4i + 3j + k$ illustrates the power of vector calculus in solving geometric problems. By calculating the cross product, we find:

$$
\mathbf{A} \times \mathbf{B} = -1 i - 3 j + 13 k
$$

and the magnitude of this vector, which corresponds to the area of the parallelogram, is:

$$
\boxed{
\sqrt{179}
}
$$

or approximately 13.379. This result exemplifies the interplay between algebra and geometry, showcasing how vector operations translate into tangible spatial measurements.

In essence, the exact area of the parallelogram formed by the given vectors is $\boxed{\sqrt{179}}$ square units, providing both a precise and an interpretable measure of this fundamental geometric figure.

Note: This detailed exploration underscores the significance of exact expressions in mathematical computation, emphasizing their importance in rigorous scientific and engineering contexts.

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