Find The Curve In The Xy-plane That Passes Through The Point (4,5) And Whose Slope At Each Point Is 3(x).

Find The Curve In The Xy-plane That Passes Through The Point (4,5) And Whose Slope At Each Point Is 3(x).
This problem involves understanding differential equations and how they relate to the geometry of curves in the xy-plane. The goal is to determine the specific curve that not only passes through a given point, (4,5), but also has a slope at any point (x, y) that is given by the function 3(x). This is a classic problem in calculus that combines concepts of derivatives, initial conditions, and solving differential equations to find the explicit form of the curve.

Understanding the Problem

Before jumping into the solution, it’s important to understand the key components of the problem:

- The curve passes through the point (4, 5).
- The slope of the curve at any point (x, y) is given by the function 3(x).
- The slope, or derivative dy/dx, is a function of x only, which simplifies the problem to a first-order ordinary differential equation (ODE).

This problem essentially asks us to find a function y = y(x) that satisfies these conditions. The slope condition dy/dx = 3(x) provides a direct way to describe how y changes as x changes, and integrating this differential equation will lead us to the explicit form of the curve.

Setting Up the Differential Equation

Given that the slope at each point is 3(x), the differential equation is:

$$
\frac{dy}{dx} = 3x
$$

This is a straightforward first-order ODE that can be solved using basic integration techniques.

Solving the Differential Equation

Integration Process

To find y as a function of x, integrate both sides of the differential equation:

$$
dy = 3x \, dx
$$

Integrating both sides:

\[

\int dy = \int 3x \, dx
\]

which simplifies to:

\[
y = \frac{3x^2}{2} + C
\]

where C is the constant of integration. The solution y = (3/2) x^2 + C represents a family of parabolas, each differing by the value of C.

Applying the Initial Condition

To determine the specific curve passing through (4, 5), apply the initial condition:

\[
x = 4, \quad y = 5
\]

Substituting into the general solution:

\[
5 = \frac{3}{2} (4)^2 + C
\]

Calculate:

\[
(4)^2 = 16
\]

so,

\[
5 = \frac{3}{2} \times 16 + C
\]

\[
5 = 24 + C
\]

Solve for C:

\[
C = 5 - 24 = -19
\]

Final Equation of the Curve

Putting it all together, the explicit equation of the curve is:

$$
\boxed{
y = \frac{3}{2} x^2 - 19
}
$$

This parabola passes through the point (4, 5) and has a slope of $3(x)$ at every point along its path.

Graphical Interpretation

Shape and Orientation

The curve $y = (3/2) x^2 - 19$ is a parabola opening upward, as the coefficient of x^2 is positive. Its vertex is at the point where the derivative $dy/dx = 3x$ equals zero, which occurs at:

$$
x = 0
$$

At $x = 0$, $y = (3/2) 0 - 19 = -19$, so the vertex is at (0, -19).

Slope at Various Points

Since the slope at any point is $3x$, it varies linearly with x:

- At $x = 0$, the slope is 0.
- At $x = 4$, the slope is $3(4) = 12$.
- At $x = -4$, the slope is $3(-4) = -12$.

This provides insight into how steeply the curve ascends or descends at different points.

Applications and Significance

Engineering and Physics

Understanding how curves behave based on their slopes is essential in various fields:

- Trajectory Planning: Designing paths that follow specific slope constraints.
- Optimization Problems: Finding minima or maxima on certain curves.
- Kinematic Analysis: Describing the motion of particles along a path with a given slope function.

Mathematical Insights

This problem demonstrates:

- How differential equations can model real-world phenomena.
- The importance of initial conditions in determining unique solutions.
- The role of integration in solving first-order ODEs.

Additional Considerations

Alternative Methods

While direct integration is the most straightforward approach here, other techniques include:

- Separable Differential Equations: Recognizing the form dy/dx = 3x as separable.
- Graphical Methods: Sketching the family of curves for various values of C and then selecting the one passing through the specific point.

Limitations and Extensions

- If the slope function depended on both x and y, the problem would be more complex, requiring different methods such as substitution or numerical solutions.
- For more complicated slope functions, techniques like integrating factors or numerical solvers may be necessary.

Conclusion

In summary, the curve in the xy-plane that passes through the point (4, 5) and has a slope at each point given by 3(x) is the parabola:

\[
\boxed{
y = \frac{3}{2} x^2 - 19
}
\]

This problem exemplifies the power of differential equations in modeling and solving problems involving curves and slopes. By integrating the slope function and applying initial conditions, we derive the explicit equation of the curve, offering both algebraic and geometric insights into its behavior. Whether in pure mathematics or applied sciences, such techniques are fundamental tools for understanding the relationships between a function and its rate of change.

Frequently Asked Questions

What is the general approach to find the curve in the XY-plane given a slope function?

To find the curve, you integrate the slope function dy/dx = 3x with respect to x, then use the initial point (4, 5) to solve for the constant of integration.

How do you determine the specific curve passing through (4, 5) with a given slope function?

You integrate dy/dx = 3x to find y = (3/2)x^2 + C, then substitute x = 4 and y = 5 to solve for C, which yields C = 5 - 6 = -1.

What is the explicit equation of the curve passing through (4, 5) with slope 3x?

The equation of the curve is $y = (3/2)x^2 - 1$.

How can I verify that the curve $y = (3/2)x^2 - 1$ passes through the point (4, 5)?

Substitute $x = 4$ into the equation: $y = (3/2)(4)^2 - 1 = (3/2)(16) - 1 = 24 - 1 = 23$. Since this does not match $y=5$, recheck the calculations; actually, the correct C after substitution should yield $y=5$, so the initial C calculation needs correction.

What is the correct constant of integration for the curve passing through (4, 5)?

Integrate $dy/dx = 3x$ to get $y = (3/2)x^2 + C$. Substituting $x=4$, $y=5$ gives $5 = (3/2)(16) + C$, so $C = 5 - 24 = -19$. Therefore, the equation is $y = (3/2)x^2 - 19$.

What is the final equation of the curve passing through (4, 5) with slope 3x?

The final equation is $y = (3/2)x^2 - 19$.

Additional Resources

Find The Curve In The XY-plane That Passes Through The Point (4,5) And Whose Slope At Each Point Is 3(x)

Understanding the problem of finding a specific curve in the XY-plane can be both intellectually stimulating and practically useful in fields such as physics, engineering, and mathematics. At the core of this problem lies the concept of differential equations, which serve as a bridge between the geometric properties of curves and their algebraic descriptions. The task at hand involves finding a curve that passes through a given point—(4, 5)—and whose slope at any point (x, y) on the curve is given by a specific function—in this case, 3(x). This presents an excellent opportunity to explore the methods of solving first-order differential equations and to appreciate the role of initial conditions in determining specific solutions.

In this article, we'll delve into the theoretical background, step-by-step solution process, interpretation of the results, and practical implications of such a problem, providing a comprehensive understanding for students, educators, and enthusiasts alike.

Understanding the Problem

What is a Differential Equation?

A differential equation is a mathematical equation that relates a function with its derivatives. It expresses how a quantity changes with respect to another—typically, how y varies with x. In the context of our problem, the differential equation states that the slope of the curve at any point (x, y) is 3(x). Mathematically, this is written as:

$$
\frac{dy}{dx} = 3x
$$

This indicates that the rate of change of y with respect to x depends solely on the value of x, making this a first-order differential equation with explicit dependence on the independent variable x.

Initial Conditions and Their Importance

The point (4, 5) serves as an initial condition, often called a boundary condition or an initial value. It specifies a particular solution out of infinitely many solutions to the differential equation, by requiring the curve to pass through this point. Properly incorporating this condition ensures the uniqueness of the solution and tailors the general solution to our specific problem.

Mathematical Approach to the Solution

Step 1: Recognize the Type of Differential Equation

The given differential equation:

$$
\frac{dy}{dx} = 3x
$$

is a straightforward first-order ordinary differential equation (ODE). It’s linear and separable, which simplifies the process of solving it.

Step 2: Separate Variables

Since the right side depends only on x, and y is not involved explicitly on the right side, we can integrate directly:

$$
dy = 3x\, dx
$$

This is a separable equation, allowing us to integrate both sides separately.

Step 3: Integrate Both Sides

Integrating both sides:

$$
\int dy = \int 3x\, dx
$$

$$
y = \frac{3}{2}x^2 + C
$$

where C is the constant of integration representing the family of all possible solutions.

Step 4: Apply the Initial Condition

Use the point (4, 5) to find the specific value of C:

$$
5 = \frac{3}{2} \times 4^2 + C
$$

$$
5 = \frac{3}{2} \times 16 + C
$$

$$
5 = 24 + C
$$

$$
C = 5 - 24 = -19
$$

Thus, the particular solution that passes through (4, 5) is:

$$
\boxed{y = \frac{3}{2}x^2 - 19}

\]

Analysis of the Solution

Graphical Interpretation

The curve $y = \frac{3}{2}x^2 - 19$ is a parabola opening upwards, shifted downward by 19 units. Its vertex is at the origin, but since the constant term is negative, the parabola intersects the y-axis at y = -19, and it passes through the point (4, 5) as per the initial condition.

Features of the Curve

- Shape: Parabolic, symmetric about the y-axis.
- Slope at any point: Given by the derivative $\frac{dy}{dx} = 3x$, which increases linearly with x.
- Passes through: (4, 5), satisfying the initial condition.

Implications in Practical Contexts

- The solution can model physical phenomena where the rate of change of a quantity depends linearly on the independent variable—such as certain acceleration profiles or economic models where growth rate depends on time.

Pros and Cons of the Approach

Pros:

- Simplicity: The differential equation is straightforward and easily integrable.
- Explicit Solution: The function $y = \frac{3}{2}x^2 - 19$ provides a clear, closed-form expression.
- Initial Condition Integration: The method ensures the solution is specific to the initial point, making it useful for modeling real-world scenarios.

Cons:

- Limited to Simple Equations: More complex differential equations (nonlinear, higher order) require advanced methods.
- Assumption of Continuity: The solution assumes the function is continuous and differentiable, which may not hold in all physical contexts.

- No Consideration of External Factors: The model only accounts for the slope defined by the differential equation, ignoring other influences.

Extensions and Related Topics

Higher-Order Differential Equations

The current problem involves a first-order, linear differential equation. Similar techniques can be extended to higher-order equations, though solutions often involve more complex methods such as characteristic equations or Laplace transforms.

Nonlinear Differential Equations

If the slope depended on y as well as x, for example, $\frac{dy}{dx} = 3x y$, the problem would become nonlinear, requiring different solution strategies like substitution or numerical methods.

Initial Value Problems (IVPs) and Boundary Value Problems (BVPs)

The initial condition used here is an example of an IVP. Such problems are fundamental in modeling physical systems where the state at a specific point is known, and the goal is to find the entire solution.

Practical Applications

Understanding and solving differential equations like this one have broad applications:

- Physics: Modeling velocity and acceleration where the rate of change depends on position.
- Economics: Growth models where the rate of change of investment depends on current values.
- Biology: Population growth models with rates depending on current population.
- Engineering: Design of control systems where response depends linearly on certain parameters.

Conclusion

Finding the curve in the XY-plane that passes through a specific point and has a given slope function exemplifies the power and elegance of differential equations. The process involves recognizing the nature of the differential equation, integrating to find the general solution, and then applying initial conditions to determine the particular solution. In this case, the solution $y = \frac{3}{2}x^2 - 19$ perfectly fits the specified initial point and slope function.

This problem not only demonstrates fundamental techniques in solving differential equations but also underscores their importance in modeling real-world phenomena. Whether in physics, engineering, or economics, the ability to derive and interpret such solutions is a valuable skill. As mathematical complexity increases, so does the richness of solutions and the methods required, making the study of differential equations a continually rewarding field.

In summary:

- The differential equation $\frac{dy}{dx} = 3x$ defines the slope at each point.
- Integrating yields the general parabola $y = \frac{3}{2}x^2 + C$.
- Applying initial conditions determines the specific curve passing through (4, 5).
- The resulting parabola is a fundamental example illustrating how initial data shapes solutions.
- Such techniques form the backbone of modeling dynamic systems across science and engineering.

Final note: Mastery of solving and interpreting differential equations opens doors to a deeper understanding of natural phenomena and the mathematical structures underlying them.

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