Find The First Five Non-zero Terms Of Power Series Representation Centered At X=0 For The Function Below.

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When working with functions in calculus, especially in the context of series expansions, it's often essential to express functions as power series centered at a particular point—in this case, at $x=0$. This process not only simplifies the analysis of functions near that point but also provides valuable insights into their behavior, derivatives, and integrals. In this article, we will explore how to find the first five non-zero terms of the power series representation centered at $x=0$ for a given function, illustrating the method step-by-step.

Understanding Power Series and Their Significance

What Is a Power Series?

A power series is an infinite sum of the form:

$$
f(x) = \sum_{n=0}^\infty a_n x^n
$$

where a_n are coefficients that depend on the function, and x is the variable. Power series are particularly useful because they can approximate complex functions near a specific point—here, at $x=0$, which is referred to as the Maclaurin series.

Why Center at $x = 0$?

Centering at $x=0$ (Maclaurin series) simplifies calculations because derivatives are evaluated at zero, making the process straightforward. Many common functions, such as exponential, sine, cosine, and logarithmic functions, have well-known Maclaurin series expansions.

Applications of Power Series Expansions

Power series are used in:

- Approximating functions near the expansion point
- Simplifying complex functions for integration and differentiation
- Solving differential equations
- Analyzing convergence and behavior of functions

Step-by-Step Method to Find the First Five Non-zero Terms

1. Identify the Function and Its Behavior at $x=0$

Begin by understanding the given function's properties at $x=0$. Check whether the function is defined at zero and whether it has derivatives of all orders at that point.

2. Compute the Derivatives at $x=0$

The Maclaurin series coefficients are obtained using:

$$
a_n = \frac{f^{(n)}(0)}{n!}
$$

where $f^{(n)}(0)$ is the n-th derivative evaluated at zero.

3. Find the Series Terms and Identify Non-zero Coefficients

Calculate derivatives step-by-step, evaluate at zero, and determine which terms are non-zero. The first five non-zero terms are those with non-zero coefficients.

4. Write the Power Series Up to the Fifth Non-zero Term

Express the sum of the terms explicitly, including the coefficients and powers of x.

5. Simplify and Present the Series

Combine like terms if possible, and clearly state the resulting series expansion.

Illustrative Example: Find the First Five Non-zero Terms of $f(x) = e^x$

Step 1: Recognize the Function

The exponential function e^x is well-understood, and its derivatives are straightforward.

Step 2: Calculate Derivatives at Zero

Since $\frac{d^n}{dx^n} e^x = e^x$, evaluating at zero gives:

$$
f^{(n)}(0) = e^0 = 1
$$

for all n.

Step 3: Find Coefficients a_n

$$
a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{n!}
$$

The coefficients are $1/n!$.

Step 4: Write the Series and Extract Non-zero Terms

The Maclaurin series for e^x is:

$$
e^x = \sum_{n=0}^\infty \frac{x^n}{n!}
$$

The first five non-zero terms are:

$$
1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}
$$

or explicitly:

$$
1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}
$$

Applying the Method to Other Functions

The process outlined above applies broadly. Let's examine a few common functions and find their first five non-zero terms.

1. Sine Function: $f(x) = \sin x$

- Derivatives at zero:
- $f(0) = 0$
- $f'(x) = \cos x \Rightarrow \cos 0 = 1$
- $f''(x) = -\sin x \Rightarrow 0$
- $f'''(x) = -\cos x \Rightarrow -1$
- $f^{(4)}(x) = \sin x \Rightarrow 0$
- $f^{(5)}(x) = \cos x \Rightarrow 1$

- Coefficients:
- $a_1 = 1/1! = 1$
- $a_3 = -1/3! = -1/6$
- $a_5 = 1/5! = 1/120$

- Series:

$$
\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}
$$

The first five non-zero terms are these three; higher-order terms follow similarly.

2. Cosine Function: $f(x) = \cos x$

- Derivatives at zero:
- $f(0) = 1$
- $f'(x) = -\sin x \Rightarrow 0$
- $f''(x) = -\cos x \Rightarrow -1$
- $f'''(x) = \sin x \Rightarrow 0$
- $f^{(4)}(x) = \cos x \Rightarrow 1$

- Coefficients:
- $a_0 = 1$
- $a_2 = -1/2! = -1/2$
- $a_4 = 1/4! = 1/24$

- Series:

$$
\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}
$$

Tips for Finding Power Series Terms Effectively

- **Identify the pattern of derivatives:** Recognize if derivatives repeat or follow a pattern, simplifying calculations.
- **Check for zero derivatives:** Some functions have derivatives that are zero at certain orders, allowing you to skip those terms.
- **Use known series expansions:** Many functions have standard Maclaurin series, which can be memorized or referenced for quick approximation.
- **Verify convergence:** Ensure the power series converges to the function in the interval of interest.

Conclusion

Finding the first five non-zero terms of a power series centered at $x=0$ is a fundamental skill in calculus, useful for approximating functions and solving various mathematical problems. The process involves calculating derivatives at zero, deriving the coefficients, and writing out the series explicitly. Whether working with exponential, sine, cosine, or other functions, understanding this method enhances your ability to analyze and interpret functions in their series form.

By practicing with different functions and recognizing patterns in derivatives, you can efficiently generate power series expansions that serve as powerful tools in both theoretical and applied mathematics. Remember, the key steps are identifying derivatives at zero, computing coefficients, and carefully constructing the series terms to capture the function's behavior near the expansion point.

Keywords: Power Series, Maclaurin Series, Series Expansion, Non-zero Terms, Function Approximation, Calculus, Derivatives, Series Coefficients, Mathematical Series, Function Analysis

Frequently Asked Questions

How do I find the first five non-zero terms of the power series for a function centered at x=0?

To find the first five non-zero terms, you typically expand the function into its Maclaurin series (power series centered at 0) by calculating derivatives at 0 and applying Taylor's formula, then identify and list the first five terms with non-zero coefficients.

What is the importance of identifying the first five non-zero terms in a power series expansion?

Identifying these terms helps understand the function's behavior near the center point, simplifies calculations, and allows for approximations of the function within the radius of convergence, especially when the initial terms provide sufficient accuracy.

Can you give an example of finding the first five non-zero terms for a common function, like e^x?

Yes. For e^x, the Maclaurin series is e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + ..., so the first five non-zero terms are 1, x, x^2/2, x^3/6, and x^4/24.

What steps should I follow if the function has zero derivatives at x=0 for initial orders?

If the derivatives at zero are zero for initial orders, look for the next non-zero derivative at higher orders, then include those terms in your series until you have identified five non-zero terms.

How does the radius of convergence affect the power series expansion centered at x=0?

The radius of convergence determines the interval around x=0 where the power series converges to the function. Within this radius, the series accurately represents the function, and understanding it helps assess how many terms are needed for a good approximation.

Are there tools or software that can assist in finding the first five non-zero terms of a power series?

Yes, mathematical software such as Wolfram Alpha, Wolfram Mathematica, Maple, and even online calculators can help compute derivatives and generate power series expansions, making it easier to find the first five non-zero terms.

Additional Resources

Find The First Five Non-zero Terms Of Power Series Representation Centered At X=0 For The Function Below

In the realm of mathematical analysis, power series serve as powerful tools for approximating functions, analyzing their behavior, and solving complex problems across various fields such as engineering, physics, and computer science. A common task involves deriving the power series expansion—also known as the Taylor series—centered at a specific point, often at zero, to facilitate easier computations and insights. This article delves into the process of finding the first five non-zero terms of a power series representation centered at $x=0$ for a given function, illustrating the methodology with detailed steps and explanations suitable for both students and practitioners.

Understanding Power Series and Their Significance

Before we proceed to the specific problem, it is vital to comprehend what a power series entails and why it is instrumental in mathematical analysis.

What Is a Power Series?

A power series is an infinite sum of the form:

$$
f(x) = \sum_{n=0}^\infty a_n (x - c)^n
$$

where:

- a_n are coefficients that depend on the function,
- c is the center of the expansion (for this discussion, $c=0$),
- x is the variable.

When the center $c=0$, the series simplifies to:

$$
f(x) = \sum_{n=0}^\infty a_n x^n
$$

Power series can approximate functions within their radius of convergence, providing a polynomial-like representation that simplifies calculations and analysis.

Why Focus on the First Few Non-zero Terms?

In practical applications, often only the initial non-zero terms of a power series are necessary to approximate a function with sufficient accuracy near the expansion point. This is especially useful for small values of x, where higher-order terms become negligible. Identifying these terms is crucial in:

- Engineering for signal processing,
- Physics for modeling phenomena,
- Numerical methods for solving differential equations.

The Core Problem: Deriving the Series for a Given Function

Suppose we are given a specific function $f(x)$. Our goal is to determine its power series expansion centered at $x=0$, and specifically, to find the first five non-zero terms.

For illustration, let’s consider the function:

$$
f(x) = \frac{1}{1 - x} + \sin x
$$

This composite function combines a rational function with a well-known trigonometric function, both of which have known power series expansions. The challenge lies in combining these series to identify the initial non-zero terms of the overall expansion.

Step-by-Step Methodology

To systematically find the first five non-zero terms, follow these steps:

1. Expand Each Component Individually

Identify the power series expansion for each component function around \(x=0 \).

a) Expansion of \(\frac{1}{1 - x} \):

This is a geometric series with ratio \(x \), valid for \(|x|<1 \):

\[
\frac{1}{1 - x} = \sum_{n=0}^\infty x^n = 1 + x + x^2 + x^3 + x^4 + \cdots
\]

The first five non-zero terms are:

\[
1 + x + x^2 + x^3 + x^4
\]

b) Expansion of \(\sin x \):

The Maclaurin series for \(\sin x \):

\[
\sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}
\]

Expanding up to the \(x^5 \) term:

\[
\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots
\]

The first three non-zero terms are:

\[
x - \frac{x^3}{6} + \frac{x^5}{120}
\]

Since we're focusing on the first five non-zero terms overall, note that these terms are of degrees 1, 3, and 5.

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2. Combine the Series

Now, sum the individual series term-by-term, aligning powers of \(x \):

\[
f(x) = \left(1 + x + x^2 + x^3 + x^4 + \cdots \right) + \left(x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots \right)

\]

Group like terms:

- Constant term (x^0): 1
- x^1: $x + x = 2x$
- x^2: x^2
- x^3: $x^3 - \frac{x^3}{6} = x^3 \left(1 - \frac{1}{6}\right) = \frac{5}{6} x^3$
- x^4: x^4
- x^5: $\frac{x^5}{120}$

Beyond x^5, terms continue, but for our purpose, we focus on the first five non-zero terms.

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3. Identify the First Five Non-zero Terms

The terms identified are:

1. 1 (constant term)
2. $2x$
3. x^2
4. $\frac{5}{6} x^3$
5. x^4

These comprise the first five non-zero terms of the power series expansion of $f(x)$ centered at $x=0$.

Generalization and Practical Insights

The process demonstrated above is adaptable to a wide range of functions. The key steps involve:

- Recognizing known series expansions for component functions,
- Ensuring the series are expanded to sufficient order,
- Combining like terms carefully,
- Identifying the initial non-zero terms to approximate the function near $x=0$.

This approach is particularly powerful when dealing with functions that are sums, products, or compositions of elementary functions with well-understood series expansions.

Applications and Implications

Deriving the initial terms of a power series expansion has numerous practical applications:

- Approximate calculations: Engineers often need to approximate complex functions with polynomials for quick computations.
- Numerical analysis: Many numerical methods rely on truncating series to a finite number of terms.
- Differential equations: Power series solutions are fundamental in solving differential equations with variable coefficients.
- Physics modeling: Small-angle approximations in mechanics and wave physics often use series expansions for simplicity.

Understanding how to find and interpret these series enables professionals across disciplines to analyze behaviors, optimize designs, and solve problems with greater clarity.

Final Remarks

Mastering the process of extracting the first five non-zero terms of a power series centered at zero enhances both theoretical understanding and practical problem-solving skills. Whether analyzing simple functions or tackling complex composite functions, this methodology provides a systematic pathway to approximate functions locally with polynomials that are easier to manipulate and interpret.

As a rule of thumb, always verify the convergence radius and the validity of the approximation within the desired domain. With practice, deriving these initial terms becomes second nature, empowering analysts and students alike to navigate the fascinating world of power series with confidence and precision.

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