Find The Point(s) Of Intersection (if Any) Of The Plane And The Line. (If An Answer Does Not Exist, Enter

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Understanding the intersection between a plane and a line is a fundamental concept in analytic geometry. Whether you're a student learning the basics of coordinate geometry or a professional dealing with spatial problems, mastering how to find the intersection points of a line and a plane is essential. This article provides a comprehensive guide to determining whether a line intersects a plane, finding the intersection point(s) when they do, and understanding cases where no intersection exists.

Introduction to Planes and Lines in 3D Space

A plane in three-dimensional space can be described mathematically by a plane equation, typically in the form:

$$ Ax + By + Cz + D = 0 $$

where (A, B, C) are the coefficients representing the normal vector to the plane, and D is a constant.

A line in space can be represented parametrically as:

$$
\begin{cases}
x = x_0 + t a \\
y = y_0 + t b \\
z = z_0 + t c
\end{cases}
$$

where (x_0, y_0, z_0) is a point on the line, (a, b, c) is the direction vector of the line, and t is a parameter.

The key problem is to determine whether the line intersects the plane, and if so, at which point(s).

Understanding the Intersection Problem

Given:

- The plane equation: $A x + B y + C z + D = 0$
- The parametric equations of the line: $x = x_0 + t a$, $y = y_0 + t b$, $z = z_0 + t c$

Our goal:

- Find the value(s) of t for which the parametric point lies on the plane.
- Determine the intersection point(s) by plugging back the value(s) of t into the parametric equations.

Step-by-Step Process to Find the Intersection Point(s)

1. Substituting the Parametric Equations into the Plane Equation

Begin by substituting x, y, z from the parametric equations into the plane equation:

$$
A(x_0 + t a) + B(y_0 + t b) + C(z_0 + t c) + D = 0
$$

Expanding this:

$$
A x_0 + A t a + B y_0 + B t b + C z_0 + C t c + D = 0
$$

Group the terms involving t:

$$
(A a + B b + C c) t + (A x_0 + B y_0 + C z_0 + D) = 0
$$

This simplifies to a linear equation in t:

$$
\textbf{Coefficient of } t: \quad M = A a + B b + C c
$$

$$
\textbf{Constant term:} \quad N = A x_0 + B y_0 + C z_0 + D
$$

Thus, the equation becomes:

$$
M t + N = 0
$$

2. Solving for t

- If $M \neq 0$:

$$
t = -\frac{N}{M}
$$

- If $M = 0$:

- If $N = 0$:

The entire line lies on the plane (the line is contained within the plane), resulting in infinitely many intersection points.

- If $N \neq 0$:

The line is parallel to the plane and does not intersect it (no solution).

3. Finding the Intersection Point(s)

- When t is determined, substitute back into the parametric equations:

$$
x = x_0 + t a,\quad y = y_0 + t b,\quad z = z_0 + t c
$$

- This gives the point of intersection:

$$
\boxed{
\left(x_0 + t a,\ y_0 + t b,\ z_0 + t c \right)
}
$$

- If infinitely many solutions exist, the line lies entirely within the plane.

Special Cases and Their Interpretations

Understanding special cases is crucial.

Case 1: Unique Intersection Point (Line Intersects the Plane at a Single Point)

- Occurs when $M \neq 0$, leading to a unique t, and consequently, a unique point of intersection.

Case 2: The Line Lies Entirely in the Plane

- Happens when $M = 0$ and $N = 0$.

- The parametric equations satisfy the plane equation for all values of t.

- The entire line is contained within the plane.

Case 3: No Intersection (Line is Parallel and Outside the Plane)

- When $M = 0$ but $N \neq 0$, the line and plane are parallel, and no point of intersection exists.

Examples to Illustrate the Process

Example 1: Line Intersects the Plane at a Single Point

Suppose:

- Plane: $2x - y + 3z - 4 = 0$
- Line: $x = 1 + t,\ y = 2 + 2t,\ z = 3 + t$

Step 1: Compute M:

$$
A a + B b + C c = 2 \times 1 + (-1) \times 2 + 3 \times 1 = 2 - 2 + 3 = 3
$$

Step 2: Compute N:

$$
A x_0 + B y_0 + C z_0 + D = 2 \times 1 + (-1) \times 2 + 3 \times 3 - 4 = 2 - 2 + 9 - 4 = 5
$$

Step 3: Find t:

$$
t = -\frac{N}{M} = -\frac{5}{3}
$$

Step 4: Find the intersection point:

$$
x = 1 + \left(-\frac{5}{3}\right) = 1 - \frac{5}{3} = -\frac{2}{3}
$$
$$
y = 2 + 2 \times \left(-\frac{5}{3}\right) = 2 - \frac{10}{3} = -\frac{4}{3}
$$
$$
z = 3 + \left(-\frac{5}{3}\right) = 3 - \frac{5}{3} = \frac{4}{3}
$$

Result:

The line intersects the plane at $\left(-\frac{2}{3}, -\frac{4}{3}, \frac{4}{3}\right)$.

Example 2: Line Lies Entirely in the Plane

Suppose:

- Plane: $x + y + z = 6$
- Line: $x = 2 + t,\ y = 1 + t,\ z = 3 - t$

Check: Is the line contained in the plane?

Calculate $A a + B b + C c$:

$$
1 \times 1 + 1 \times 1 + 1 \times (-1) = 1 + 1 - 1 = 1 \neq 0
$$

So, not parallel; proceed.

Find N:

\[

$$
A x_0 + B y_0 + C z_0 + D = (1)(2) + (1)(1) + (1)(3) - 6 = 2 + 1 + 3 - 6 = 0
$$

Since $M = 1 \neq 0$, the line intersects the plane at a point.

But suppose instead we have:

- Line: $x=1 + 2t,\ y=2 + 2t,\ z=3 + 2t$

Calculate M:

$$
1 \times 2 + 1 \times 2 + 1 \times 2 = 2 + 2 + 2 = 6 \neq 0
$$

Find N:

$$
1 \times 1 + 1 \times 2 + 1 \times 3 - 6 = 1 + 2 + 3 - 6 = 0
$$

$$t = -\frac{N}{M} = 0$$

Substitute $t=0$:

$$
x=1,\ y=2,\ z=3
$$

Check if this point lies on the plane:

$$
1 + 2 + 3 = 6 \quad \checkmark
$$

Thus, the line intersects the plane at $(1, 2, 3)$.

When No Intersection Exists: Parallel Lines and Planes

Frequently Asked Questions

How do I find the point of intersection between a plane and a line in 3D space?

To find the intersection, substitute the parametric equations of the line into the equation of the plane and solve for the parameter. The resulting point gives the intersection point if it exists.

What does it mean if the system of equations for the plane and line has no solution?

It means the line and the plane do not intersect; they are parallel and separate, so there is no point of intersection.

How can I determine if a line lies entirely within a plane?

If every point on the line satisfies the plane's equation, then the line lies entirely within the plane. This can be checked by substituting the line's parametric equations into the plane's equation.

What if the line is contained within the plane—how is the intersection point described?

In this case, every point on the line is an intersection point, so there are infinitely many points of intersection.

What is the significance of the determinant when finding the intersection point?

The determinant helps determine whether the system of equations has a unique solution (single intersection), no solution (parallel, no intersection), or infinitely many solutions (line lies within the plane).

Can a line intersect a plane at more than one point?

No, a line can intersect a plane at most at one point unless it lies entirely within the plane, in which case there are infinitely many intersection points.

If I can't find an intersection point, what should I do?

If no intersection point exists, the equations are inconsistent or represent parallel entities. Enter 'No intersection' or leave the answer blank as per instructions.

Additional Resources

Intersection of a Plane and a Line: An In-Depth Exploration

Understanding the geometric relationship between lines and planes is fundamental to numerous fields—from architecture and engineering to computer graphics and robotics. One of the core concepts in this realm is determining whether a line intersects a plane, and if so, at what point(s). This article provides a comprehensive guide to finding the point(s) of intersection between a plane and a line, exploring methods, potential complications, and practical applications.

Introduction to Planes and Lines in Space

Before delving into intersection calculations, it's essential to understand the basic definitions and representations of lines and planes in three-dimensional space.

Lines in Space

A line in three-dimensional space can be represented parametrically as:

$$
\mathbf{r} = \mathbf{r}_0 + t \mathbf{v}
$$

where:
- $\mathbf{r}_0 = (x_0, y_0, z_0)$ is a point through which the line passes.
- $\mathbf{v} = (a, b, c)$ is the direction vector of the line.
- t is a real parameter that varies over all real numbers.

This representation indicates that for each value of t, the point $\mathbf{r}$ lies on the line.

Planes in Space

A plane can be described by a point and a normal vector or by an equation in the form:

$$
A x + B y + C z + D = 0
$$

where:
- (A, B, C) is the normal vector $\mathbf{n}$ to the plane.
- D is a scalar constant.

Any point (x, y, z) satisfying this equation lies on the plane.

Determining the Intersection: Theoretical Foundations

The goal is to find the point(s) where the line and the plane meet. This involves solving the equations simultaneously.

Methodology Overview

The general process involves:
1. Expressing the line parametrically.
2. Substituting the parametric equations into the plane equation.
3. Solving for the parameter t.
4. Using the value(s) of t to find the corresponding point(s).

Step-by-Step Procedure for Finding Intersection Points

Step 1: Write the Parametric Equations of the Line

Suppose the line is given by:

$$
\mathbf{r}(t) = (x_0 + a t, y_0 + b t, z_0 + c t)
$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$ and $\mathbf{v} = (a, b, c)$.

Step 2: Write the Plane Equation

Assuming the plane is described by:

$$
A x + B y + C z + D = 0
$$

Step 3: Substitute the Line Equations into the Plane Equation

Substitute $x = x_0 + a t$, $y = y_0 + b t$, and $z = z_0 + c t$:

$$
A (x_0 + a t) + B (y_0 + b t) + C (z_0 + c t) + D = 0
$$

Expanding:

$$
A x_0 + A a t + B y_0 + B b t + C z_0 + C c t + D = 0
$$

Grouping terms:

$$
(A a + B b + C c) t + (A x_0 + B y_0 + C z_0 + D) = 0
$$

This is a linear equation in t:

$$
\boxed{ \left(A a + B b + C c \right) t = - \left(A x_0 + B y_0 + C z_0 + D \right) }
$$

Analyzing the Results: When Does an Intersection Exist?

The nature of the solution depends on the coefficients:

Case 1: Unique Intersection Point

If $A a + B b + C c \neq 0$, then:

$$
t = - \frac{A x_0 + B y_0 + C z_0 + D}{A a + B b + C c}
$$

which provides a unique real value for t. Plugging this back into the parametric equations yields the unique point of intersection:

$$
\boxed{
\begin{aligned}
x &= x_0 + a t \\
y &= y_0 + b t \\
z &= z_0 + c t
\end{aligned}
}
$$

This is the point where the line pierces the plane.

Case 2: No Intersection (Parallel and Not in the Plane)

If $A a + B b + C c = 0$ but:

$$
A x_0 + B y_0 + C z_0 + D \neq 0
$$

Then the line is parallel to the plane but does not lie on it; thus, no intersection exists.

Case 3: Infinite Solutions (Line Lies Entirely in the Plane)

If both:

$$
A a + B b + C c = 0
$$
and
$$
A x_0 + B y_0 + C z_0 + D = 0
$$

then every point on the line satisfies the plane equation. The line lies entirely within the plane, implying infinitely many intersection points.

Special Considerations and Practical Examples

Example 1: Intersecting Line and Plane

Suppose the line is given by:

\[

$$
\mathbf{r}(t) = (1, 2, 3) + t (4, -1, 2)
$$

and the plane is:

$$
2x - y + z - 5 = 0
$$

Solution:

1. Substitute parametric equations into plane:
$$
2(1 + 4t) - (2 - t) + (3 + 2t) - 5 = 0
$$

2. Simplify:
$$
2 + 8t - 2 + t + 3 + 2t - 5 = 0
$$
$$
(8t + t + 2t) + (2 - 2 + 3 - 5) = 0
$$
$$
11t + (-2) = 0
$$
$$
11t = 2
$$
$$
t = \frac{2}{11}
$$

3. Find point:
$$

$$
x = 1 + 4 \times \frac{2}{11} = 1 + \frac{8}{11} = \frac{19}{11}
$$

$$
y = 2 - \frac{2}{11} = \frac{22}{11} - \frac{2}{11} = \frac{20}{11}
$$

$$
z = 3 + 2 \times \frac{2}{11} = 3 + \frac{4}{11} = \frac{37}{11}
$$

Intersection Point:

$$
\boxed{
\left(\frac{19}{11}, \frac{20}{11}, \frac{37}{11} \right)
}
$$

Example 2: Line Parallel to Plane, No Intersection

Line: $\mathbf{r}(t) = (0, 0, 0) + t (1, 1, 0)$

Plane: $x + y + z + 1 = 0$

Check:

$$
A a + B b + C c = 1 \times 1 + 1 \times 1 + 0 \times 0 = 2 \neq 0
$$

Compute:

$$
A x_0 + B y_0 + C z_0 + D = 1 \times 0 + 1 \times 0 + 1 \times 0 + 1 = 1 \neq 0
$$

Since the dot product is not zero, the line is neither parallel nor lying in the plane, so an intersection exists. Wait, but in this case, the direction vector is (1,1,0), and the plane's normal is (1,1,1). The dot product is 2, indicating the line is not parallel; thus, there will be a unique intersection.

Now, if instead the line's direction vector was orthogonal to the plane's normal, say (1, -1, 0), then:

$$
1 \times 1 + (-1) \times 1 + 0 \times 0 = 0
$$

and the line's point: (0,0,0):

\[
A x_0 + B

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