Find The Solution To The Linear System Of Differential Equations {x' = 11x + 24y Y' = -3x - 6y Satisfying

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Introduction

The system of linear differential equations presented:

\[
\begin{cases}
x' = 11x + 24y \\
y' = -3x - 6y
\end{cases}
\]

represents a coupled set of equations where the derivatives of \(x(t) \) and \(y(t) \) depend linearly on both variables. Solving such systems involves several steps, including rewriting the system in matrix form, finding the eigenvalues and eigenvectors of the coefficient matrix, and then constructing the general solution. This article provides a comprehensive guide to solving this particular system, detailing each step and explaining the reasoning behind it.

Matrix Representation of the System

Converting the system into matrix form simplifies the process of solving:

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix}
=
\begin{bmatrix}
11 & 24 \\
-3 & -6
\end{bmatrix}
\begin{bmatrix}

x \\
y
\end{bmatrix}
\]

Define the vector \(\mathbf{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}\) and the coefficient matrix:

\[
A = \begin{bmatrix}
11 & 24 \\
-3 & -6
\end{bmatrix}
\]

The system becomes:

\[
\mathbf{X}'(t) = A \mathbf{X}(t)
\]

The primary objective is to find the general solution:

\[
\mathbf{X}(t) = \text{(general solution involving eigenvalues and eigenvectors)}
\]

which will then be expressed explicitly in terms of \(x(t)\) and \(y(t)\).

Finding Eigenvalues of the Coefficient Matrix

The solution hinges on the eigenvalues of matrix \(A\). The eigenvalues \(\lambda\) are obtained by solving the characteristic equation:

\[
\det(A - \lambda I) = 0
\]

where \(I\) is the identity matrix:

\[
A - \lambda I =
\begin{bmatrix}
11 - \lambda & 24 \\
-3 & -6 - \lambda
\end{bmatrix}
\]

The determinant:

$$
(11 - \lambda)(-6 - \lambda) - (24)(-3) = 0
$$

Calculating the determinant:

$$
(11 - \lambda)(-6 - \lambda) + 72 = 0
$$

Expanding:

$$
(11)(-6) + 11(-\lambda) + (-6)(- \lambda) + \lambda^2 + 72 = 0
$$

$$
-66 - 11\lambda + 6\lambda + \lambda^2 + 72 = 0
$$

Simplify:

$$
\lambda^2 - 5\lambda + 6 = 0
$$

This quadratic simplifies the eigenvalues calculation. Factoring:

$$
(\lambda - 2)(\lambda - 3) = 0
$$

Thus, the eigenvalues are:

$$
\boxed{
\lambda_1 = 2, \quad \lambda_2 = 3
}
$$

These eigenvalues indicate the nature of the system's solutions, such as whether they are exponential growth, decay, or saddle points.

Finding Eigenvectors Corresponding to Each Eigenvalue

Eigenvector for $\lambda_1 = 2$

Solve:

$$
(A - 2I) \mathbf{v} = 0
$$

which is:

$$
\begin{bmatrix}
11 - 2 & 24 \\
-3 & -6 - 2
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
= 0
$$

Simplify the matrix:

$$
\begin{bmatrix}
9 & 24 \\
-3 & -8
\end{bmatrix}
$$

Set up the system:

$$
\begin{cases}
9v_1 + 24v_2 = 0 \\
-3v_1 - 8v_2 = 0
\end{cases}
$$

From the first equation:

$$
9v_1 = -24v_2 \Rightarrow v_1 = -\frac{24}{9} v_2 = -\frac{8}{3} v_2
$$

Choosing $v_2 = 1$ for simplicity:

$$
v^{(1)} = \begin{bmatrix} -\frac{8}{3} \\ 1 \end{bmatrix}
$$

or, clearing fractions:

$$
v^{(1)} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}
$$

Eigenvector for $\lambda_2 = 3$

Similarly, solve:

$$
(A - 3I) \mathbf{v} = 0
$$

which becomes:

$$
\begin{bmatrix}
11 - 3 & 24 \\
-3 & -6 - 3
\end{bmatrix}
=
\begin{bmatrix}
8 & 24 \\
-3 & -9
\end{bmatrix}
$$

The system:

$$
\begin{cases}
8v_1 + 24v_2 = 0 \\
-3v_1 - 9v_2 = 0
\end{cases}
$$

From the first:

$$
8v_1 = -24v_2 \Rightarrow v_1 = -3 v_2
$$

Set $v_2 = 1$:

$$
v^{(2)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}
$$

Thus, the eigenvectors are:

$$
\boxed{
v^{(1)} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}, \quad
v^{(2)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}
}
$$

Constructing the General Solution

The general solution to the system is a linear combination of the eigenvector solutions:

$$
\mathbf{X}(t) = c_1 e^{\lambda_1 t} \mathbf{v}^{(1)} + c_2 e^{\lambda_2 t} \mathbf{v}^{(2)}
$$

Substituting the eigenvalues and eigenvectors:

$$
\mathbf{X}(t) = c_1 e^{2t} \begin{bmatrix} -8 \\ 3 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} -3 \\ 1 \end{bmatrix}
$$

which gives explicit expressions for $x(t)$ and $y(t)$:

$$
x(t) = -8 c_1 e^{2t} - 3 c_2 e^{3t}
$$
$$
y(t) = 3 c_1 e^{2t} + c_2 e^{3t}
$$

The constants c_1 and c_2 are determined based on initial conditions.

Applying Initial Conditions

Suppose initial conditions are given:

$$
x(0) = x_0, \quad y(0) = y_0
$$

Plugging in $t=0$:

$$
x(0) = -8 c_1 - 3 c_2 = x_0
$$

$$
y(0) = 3 c_1 + c_2 = y_0
$$

This forms a system of equations:

$$
\begin{cases}
-8 c_1 - 3 c_2 = x_0 \\
3 c_1 + c_2 = y_0
\end{cases}
$$

which can be solved for c_1 and c_2:

1. From the second:

$$
c_2 = y_0 - 3 c_1
$$

2. Substitute into the first:

$$
-8 c_1 - 3 (y_0 - 3 c_1) = x_0
$$

$$
-8 c_1 - 3 y_0 + 9 c_1 = x_0
$$

$$
(-8 c_1 + 9 c_1) - 3 y_0 = x_0
$$

$$
c_1 - 3 y_0 = x_0
$$

$$
c_1 = x_0 + 3 y_0
$$

Then,

$$
c_2 = y_0 - 3 c_1 = y_0 - 3 (x_0 + 3 y_0) = y_0 - 3 x_0 - 9 y_0 = -3 x_0
$$

Frequently Asked Questions

How do I find the general solution to the linear system {x' = 11x + 24y, y' = -3x - 6y}?

To find the general solution, write the system in matrix form and determine the eigenvalues and eigenvectors of the coefficient matrix. The solutions are linear combinations of exponential functions based on these eigenvalues and eigenvectors.

What are the steps to solve the linear differential system {x' = 11x + 24y, y' = -3x - 6y}?

First, write the system as a matrix equation, find the eigenvalues by solving the characteristic equation, then find the corresponding eigenvectors. Use these to construct the general solution, which involves exponential functions multiplied by constants determined by initial conditions.

How can I verify that a particular solution satisfies the system {x' = 11x + 24y, y' = -3x - 6y}?

Substitute the proposed solution into both differential equations to check if both sides are equal. If they hold true, the function is a valid solution satisfying the system.

Are the solutions to the system {x' = 11x + 24y, y' = -3x - 6y} stable?

Yes, stability depends on the eigenvalues of the coefficient matrix. If all eigenvalues have negative real parts, the system is stable; otherwise, it may be unstable. Calculating eigenvalues will determine the stability.

What is the significance of eigenvalues in solving the system {x' = 11x + 24y, y' = -3x - 6y}?

Eigenvalues help determine the behavior of the solutions, such as whether they grow, decay, or oscillate. They are essential in constructing the general solution and understanding the system's dynamics.

Additional Resources

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Introduction

In the realm of differential equations, systems of linear differential equations serve as foundational tools for modeling a multitude of real-world phenomena—from electrical circuits and mechanical systems to population dynamics and economic models. Among these, linear systems with constant coefficients are the most well-understood and manageable, owing to their algebraic structure which lends itself to systematic solution techniques.

This article delves into one such system:

$$
\begin{cases}
x' = 11x + 24y \\
y' = -3x - 6y
\end{cases}
$$

Our goal is to find the general solution to this system and explore methods to analyze and interpret its behavior. We will walk through the process step-by-step, starting with the foundational concepts of linear systems, progressing to eigenvalue and eigenvector analysis, and finally discussing the nature of the solutions within the phase space.

Understanding the System of Differential Equations

The Nature of Linear Systems

A system of linear differential equations with constant coefficients can be expressed in matrix form as:

$$
\mathbf{x}' = A \mathbf{x}
$$

where:

- $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$,
- A is a constant 2×2 matrix.

In our case,

$$
A = \begin{bmatrix} 11 & 24 \\ -3 & -6 \end{bmatrix}
$$

This form allows us to apply linear algebra techniques—particularly eigenvalue and eigenvector analysis—to find solutions systematically.

Significance of the System

The given system models the interplay between two variables, $x(t)$ and $y(t)$. The coefficients encode how each variable influences the other over time. Understanding the solution entails analyzing the properties of matrix A—specifically its eigenvalues and eigenvectors—which determine the qualitative behavior of the solutions.

Eigenvalues and Eigenvectors: The Cornerstone of the Solution

Computing the Eigenvalues

The eigenvalues λ satisfy the characteristic equation:

$$
\det(A - \lambda I) = 0
$$

where I is the identity matrix. For matrix A:

$$
A - \lambda I = \begin{bmatrix} 11 - \lambda & 24 \\ -3 & -6 - \lambda \end{bmatrix}
$$

Calculating the determinant:

$$
(11 - \lambda)(-6 - \lambda) - (-3)(24) = 0
$$

Expanding:

$$
(11)(-6 - \lambda) - \lambda(-6 - \lambda) + 72 = 0
$$

which simplifies to:

$$
-66 - 11\lambda + 6\lambda + \lambda^2 + 72 = 0
$$

Further combining like terms:

$$
\lambda^2 - 5\lambda + 6 = 0
$$

\]

This quadratic factors as:

\[
(\lambda - 2)(\lambda - 3) = 0
\]

Thus, the eigenvalues are:

\[
\boxed{
\lambda_1 = 2, \quad \lambda_2 = 3
}
\]

Computing Eigenvectors

For each eigenvalue, we find the corresponding eigenvector \(\mathbf{v}\) satisfying:

\[
(A - \lambda I)\mathbf{v} = \mathbf{0}
\]

Eigenvector for \(\lambda_1 = 2\)

Substituting \(\lambda = 2\):

\[
A - 2I = \begin{bmatrix} 11 - 2 & 24 \\ -3 & -6 - 2 \end{bmatrix} = \begin{bmatrix} 9 & 24 \\ -3 & -8 \end{bmatrix}
\]

Solve:

\[
\begin{bmatrix} 9 & 24 \\ -3 & -8 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

From the first row:

\[
9v_1 + 24v_2 = 0 \Rightarrow v_1 = -\frac{24}{9}v_2 = -\frac{8}{3}v_2
\]

Choosing \(v_2 = 1\):

$$
v^{(1)} = \begin{bmatrix} -\frac{8}{3} \\ 1 \end{bmatrix}
$$

Alternatively, for simplicity, multiply through by 3:

$$
v^{(1)} = \begin{bmatrix} -8 \\ 3 \end{bmatrix}
$$

Eigenvector for $\lambda_2 = 3$

Substituting $\lambda = 3$:

$$
A - 3I = \begin{bmatrix} 11 - 3 & 24 \\ -3 & -6 - 3 \end{bmatrix} = \begin{bmatrix} 8 & 24 \\ -3 & -9 \end{bmatrix}
$$

Solve:

$$
8v_1 + 24v_2 = 0 \Rightarrow v_1 = -3v_2
$$

Choosing $v_2 = 1$:

$$
v^{(2)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}
$$

Constructing the General Solution

Theoretical Framework

The general solution to a linear system with distinct real eigenvalues is a linear combination of exponential terms associated with each eigenvector:

$$
\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}^{(1)} + c_2 e^{\lambda_2 t} \mathbf{v}^{(2)}
$$

where c_1, c_2 are arbitrary constants determined by initial conditions.

Explicit Solution

Using the eigenvalues and eigenvectors found:

$$
\boxed{
\mathbf{x}(t) = c_1 e^{2t} \begin{bmatrix} -8 \\ 3 \end{bmatrix} + c_2 e^{3t}
\begin{bmatrix} -3 \\ 1 \end{bmatrix}
}
$$

which can be written component-wise as:

$$
\begin{cases}
x(t) = -8 c_1 e^{2t} - 3 c_2 e^{3t} \\
y(t) = 3 c_1 e^{2t} + c_2 e^{3t}
\end{cases}
$$

Analyzing the Solution: Behavior and Stability

Qualitative Behavior

The signs and magnitudes of eigenvalues dictate the nature of the system's trajectories:

- Eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ are both positive, indicating the solutions grow exponentially as $t \to \infty$.

- The dominant behavior for large t is driven by the larger eigenvalue ($\lambda=3$), leading to solutions that tend toward infinity, characteristic of an unstable node.

Phase Portraits

Plotting the trajectories in the phase space (x, y) reveals the flow of the system:

- Since both eigenvalues are positive, trajectories emanate from the origin and diverge outward along directions dictated by the eigenvectors.

- The eigenvector $\mathbf{v}^{(2)} = [-3, 1]$ corresponds to the faster-growing mode.

- The eigenvector $\mathbf{v}^{(1)} = [-8, 3]$ corresponds to a less dominant direction but still exhibits exponential growth.

Stability Considerations

- The system is unstable because solutions tend to infinity over time.

- No oscillatory behavior is present because eigenvalues are real and positive.

- The phase space features trajectories moving away from the origin, with the shape determined by the eigenvectors.

Initial Conditions and Particular Solutions

Incorporating Initial Conditions

Suppose the system has initial conditions:

$$
x(0) = x_0, \quad y(0) = y_0
$$

The constants c_1 and c_2 can be determined by solving:

$$
\begin{cases}
x_0 = -8 c_1 - 3 c_2 \\
y_0 = 3 c_1 + c_2
\end{cases}
$$

which simplifies to a system:

$$
\begin{bmatrix}
-8 & -3 \\
3 & 1
\end{bmatrix}
\begin{bmatrix}
c_1 \\ c_2
\end{bmatrix}
=
\begin{bmatrix}
x_0 \\ y_0
\end{bmatrix}
$$

The solutions for c_1 and c_2 are:

\[
c_1 = \frac{-x_0 - 3 y_0}{5}, \quad c_2 = \frac{8 y_0

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