Let T: R3 R3 Be Defined By T(x,y,z) = (x + Y,x - Y - Z, X +z). (A) Show That T Is A Matrix Transformation

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Introduction

In linear algebra, understanding whether a given function is a matrix transformation is fundamental. Matrix transformations, also known as linear transformations, are functions between vector spaces that preserve the operations of vector addition and scalar multiplication. Demonstrating that a specific function from R^3 to R^3 is a matrix transformation involves verifying these properties or expressing the transformation as a matrix multiplication.

This article aims to methodically show that the transformation T defined by

$$T(x, y, z) = (x + y, \; x - y - z, \; x + z)$$

is indeed a matrix transformation. We will explore the underlying concepts, verify the linearity properties, and explicitly find the matrix representation of T.

Understanding the Transformation T

The given transformation is:

$$
T(x, y, z) = (x + y, \; x - y - z, \; x + z)
$$

where $(x, y, z) \in \mathbb{R}^3$.

This transformation takes a vector in three-dimensional space and maps it to another vector in the same space by applying linear combinations of the original components.

Why Is It Important to Show T Is a Matrix Transformation?

- Linear transformations are functions $T: \mathbb{R}^n \to \mathbb{R}^m$ satisfying two properties:

1. Additivity: $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$
2. Homogeneity: $T(c\mathbf{v}) = c T(\mathbf{v})$

- Any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be represented by an $m \times n$ matrix A such that:

$$
T(\mathbf{x}) = A \mathbf{x}
$$

- Demonstrating that T is a matrix transformation involves confirming these properties or constructing such a matrix explicitly.

Step 1: Recall the Definition of a Linear Transformation

A function $T: \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation if for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and all scalars $c \in \mathbb{R}$:

1. Additivity:

$$
T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})
$$

2. Homogeneity:

$$
T(c \mathbf{v}) = c T(\mathbf{v})
$$

To verify these, we can use the functional form of T or directly check the properties.

Step 2: Express T in Terms of Components

Let $\mathbf{x} = (x, y, z)$, then:

$$
T(x, y, z) = (x + y, \; x - y - z, \; x + z)
$$

We can think of T as acting on the vector $(x, y, z)^T$, producing a new vector.

Step 3: Verify Linearity Properties

Additivity:

Let $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$.

Compute:

$$
T(\mathbf{u} + \mathbf{v}) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)
$$

$$
= ((x_1 + x_2) + (y_1 + y_2), \; (x_1 + x_2) - (y_1 + y_2) - (z_1 + z_2), \; (x_1 + x_2) + (z_1 + z_2))
$$

which simplifies to:

$$
= (x_1 + y_1 + x_2 + y_2, \; x_1 - y_1 - z_1 + x_2 - y_2 - z_2, \; x_1 + z_1 + x_2 + z_2)
$$

On the other hand:

$$
T(\mathbf{u}) + T(\mathbf{v}) = (x_1 + y_1, \; x_1 - y_1 - z_1, \; x_1 + z_1) + (x_2 + y_2, \; x_2 - y_2 - z_2, \; x_2 + z_2)
$$

which sums to:

$$
(x_1 + y_1 + x_2 + y_2, \; x_1 - y_1 - z_1 + x_2 - y_2 - z_2, \; x_1 + z_1 + x_2 + z_2)
$$

Since both expressions are identical, the additivity property holds.

Homogeneity:

Let $c \in \mathbb{R}$ and $\mathbf{v} = (x, y, z)$. Then:

$$
T(c \mathbf{v}) = T(c x, c y, c z) = (c x + c y, c x - c y - c z, c x + c z)
$$

which simplifies to:

$$
c (x + y, x - y - z, x + z) = c T(\mathbf{v})
$$

Thus, homogeneity is verified.

Conclusion:

Since T satisfies both properties, T is a linear transformation.

Step 4: Constructing the Matrix Representation of T

Having confirmed the linearity, the next step is to find the matrix A such that:

$$
T(\mathbf{x}) = A \mathbf{x}
$$

where $\mathbf{x} = (x, y, z)^T$.

Recall that the matrix A acts on the standard basis vectors:

$$
\mathbf{e}_1 = (1, 0, 0)^T, \quad \mathbf{e}_2 = (0, 1, 0)^T, \quad \mathbf{e}_3 = (0, 0, 1)^T
$$

and

$$
A = [T(\mathbf{e}_1) \quad T(\mathbf{e}_2) \quad T(\mathbf{e}_3)]
$$

Step 5: Find the Images of the Basis Vectors

Calculate:

1. $T(\mathbf{e}_1) = T(1, 0, 0)$:

$$
= (1 + 0, 1 - 0 - 0, 1 + 0) = (1, 1, 1)
$$

2. $T(\mathbf{e}_2) = T(0, 1, 0)$:

$$

$$
= (0 + 1, 0 - 1 - 0, 0 + 0) = (1, -1, 0)
$$

3. $T(\mathbf{e}_3) = T(0, 0, 1)$:

$$
= (0 + 0, 0 - 0 - 1, 0 + 1) = (0, -1, 1)
$$

Step 6: Form the Transformation Matrix

Construct matrix A with these images as columns:

$$
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & -1 & -1 \\
1 & 0 & 1
\end{bmatrix}
$$

Thus, the matrix representation of T is:

$$
\boxed{
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & -1 & -1 \\
1 & 0 & 1
\end{bmatrix}
}
$$

Final Verification:

To confirm, for any vector $\mathbf{x} = (x, y, z)^T$:

$$
A \mathbf{x} = \begin{bmatrix}
1 & 1 & 0 \\
1 & -1 & -1 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x \\ y \\ z
\end{bmatrix}
= \begin{bmatrix}
$$

x + y \\
x - y - z \\
x + z
\end{bmatrix} = T(x, y, z)
\]

which matches the original transformation.

Summary and Conclusion

- The transformation \(T: \mathbb{R}^3 \to \mathbb{R}^3 \), defined by \(T(x, y, z) = (x + y, x - y - z, x + z) \), is a linear

Frequently Asked Questions

What does it mean for T: R^3 → R^3 to be a matrix transformation?

It means that T is a linear transformation that can be represented as multiplication by a 3x3 matrix, satisfying additivity and scalar multiplication properties.

How do you verify that T(x, y, z) = (x + y, x - y - z, x + z) is a linear transformation?

By checking if T satisfies linearity, i.e., T(u + v) = T(u) + T(v) and T(cu) = cT(u) for all u, v in R^3 and scalars c.

What are the steps to find the matrix representation of T?

Apply T to the standard basis vectors e1, e2, e3, and use the results as columns of the matrix.

What is T(e1) where e1 = (1, 0, 0)?

T(1, 0, 0) = (1 + 0, 1 - 0 - 0, 1 + 0) = (1, 1, 1).

What is T(e2) where e2 = (0, 1, 0)?

T(0, 1, 0) = (0 + 1, 0 - 1 - 0, 0 + 0) = (1, -1, 0).

What is T(e3) where e3 = (0, 0, 1)?

T(0, 0, 1) = (0 + 0, 0 - 0 - 1, 0 + 1) = (0, -1, 1).

How is the matrix of T constructed from the images of basis vectors?

The columns of the matrix are T(e1), T(e2), and T(e3), arranged in that order.

What is the matrix representation of T?

The matrix is
[[1, 1, 0],
[1, -1, -1],
[1, 0, 1]]

Why does the existence of a matrix representation confirm T is a linear transformation?

Because any linear transformation from R^3 to R^3 can be expressed as multiplication by a matrix, and T has been shown to satisfy linearity with an explicit matrix.

Can T be expressed as a matrix multiplication T(v) = A v? If so, what is A?

Yes, T can be expressed as T(v) = A v, where A is the matrix
[[1, 1, 0],
[1, -1, -1],
[1, 0, 1]].

Additional Resources

Let T: $\mathbb{R}^3 \to \mathbb{R}^3$ be Defined by T(x, y, z) = (x + y, x - y - z, x + z).

In the realm of linear algebra, understanding whether a given transformation qualifies as a matrix transformation is fundamental. This not only provides insight into the structure and properties of the transformation but also facilitates practical computations and applications across various scientific and engineering disciplines. Here, we explore the transformation T: $\mathbb{R}^3 \to \mathbb{R}^3$ defined by T(x, y, z) = (x + y, x - y - z, x + z), with the goal of demonstrating that T is indeed a matrix transformation. This analysis will be comprehensive, delving into the underlying principles, step-by-step proofs, and implications.

Understanding the Concept of a Matrix Transformation

What Is a Matrix Transformation?

At its core, a matrix transformation (also known as a linear transformation) is a function $T: \mathbb{R}^n \to \mathbb{R}^m$ that preserves vector addition and scalar multiplication. Formally, a function T is a linear transformation if, for all vectors u, v in $\mathbb{R}^n$ and all scalars c in $\mathbb{R}$, the following two properties hold:

1. Additivity: $T(u + v) = T(u) + T(v)$
2. Homogeneity: $T(cu) = cT(u)$

These properties ensure that the transformation respects the vector space structure, making it representable via matrix multiplication. Specifically, for a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$, there exists an m×n matrix A such that:

$$ T(\mathbf{x}) = A \mathbf{x} $$

for every vector x in $\mathbb{R}^n$.

Key Point: The existence of such a matrix A is both necessary and sufficient for T to be a linear transformation. Conversely, any transformation defined by matrix multiplication is inherently linear.

Decomposing the Transformation T

Explicit Definition of T

The transformation T is given as:

$$ T(x, y, z) = (x + y,\, x - y - z,\, x + z) $$

This mapping takes a vector (x, y, z) in $\mathbb{R}^3$ and produces another vector in $\mathbb{R}^3$ with components derived via linear combinations of the original components.

Intuitive Analysis

At first glance, T appears to combine the variables x, y, z through addition and subtraction. Because these operations are linear (no products or nonlinear functions), T is a candidate for a linear transformation. To confirm this rigorously, we need to examine whether T satisfies the properties of additivity and homogeneity, and whether it can be expressed as a matrix-vector product.

Proving That T Is a Matrix Transformation

Step 1: Demonstrate Linearity via Additivity and Homogeneity

a) Additivity

Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be vectors in $\mathbb{R}^3$. Then,

\[T(\mathbf{u} + \mathbf{v}) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \]

Applying the definition of T:

\[
\begin{aligned}
T(\mathbf{u} + \mathbf{v}) &= \big((x_1 + x_2) + (y_1 + y_2),\, (x_1 + x_2) - (y_1 + y_2) - (z_1 + z_2),\, (x_1 + x_2) + (z_1 + z_2) \big) \\
&= \big(x_1 + y_1 + x_2 + y_2,\, x_1 - y_1 - z_1 + x_2 - y_2 - z_2,\, x_1 + z_1 + x_2 + z_2 \big)
\end{aligned}
\]

Now, compute T(u) + T(v):

\[
T(\mathbf{u}) + T(\mathbf{v}) = \big(x_1 + y_1,\, x_1 - y_1 - z_1,\, x_1 + z_1 \big) + \big(x_2 + y_2,\, x_2 - y_2 - z_2,\, x_2 + z_2 \big)
\]

Adding component-wise:

\[
\begin{aligned}
& (x_1 + y_1) + (x_2 + y_2) = x_1 + y_1 + x_2 + y_2 \\
& (x_1 - y_1 - z_1) + (x_2 - y_2 - z_2) = x_1 - y_1 - z_1 + x_2 - y_2 - z_2 \\
& (x_1 + z_1) + (x_2 + z_2) = x_1 + z_1 + x_2 + z_2

\end{aligned}
\]

This matches the expression obtained for T(u + v), confirming additivity.

b) Homogeneity

Let c be a scalar and u = (x, y, z). Then,

\[
T(c \mathbf{u}) = T(c x, c y, c z)
\]

Applying T:

\[
\big(c x + c y,\, c x - c y - c z,\, c x + c z \big) = c (x + y,\, x - y - z,\, x + z)
\]

which is exactly c T(u), confirming homogeneity.

Since both properties hold, T is linear.

Step 2: Express T as a Matrix-Vector Multiplication

Given the linearity, T can be represented by a matrix A such that:

\[T(\mathbf{x}) = A \mathbf{x} \]

where x is the vector \(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\).

To find A, observe how T acts on the standard basis vectors:

- For \(\mathbf{e}_1 = (1, 0, 0)\):

\[
T(1, 0, 0) = (1 + 0,\, 1 - 0 - 0,\, 1 + 0) = (1, 1, 1)
\]

- For \(\mathbf{e}_2 = (0, 1, 0)\):

\[
T(0, 1, 0) = (0 + 1,\, 0 - 1 - 0,\, 0 + 0) = (1, -1, 0)
\]

- For \(\mathbf{e}_3 = (0, 0, 1)\):

$$
T(0, 0, 1) = (0 + 0,\, 0 - 0 - 1,\, 0 + 1) = (0, -1, 1)
$$

Constructing the matrix A with these images as columns:

$$
A = \begin{bmatrix}
| & | & | \\
T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \\
| & | & |
\end{bmatrix} = \begin{bmatrix}
1 & 1 & 0 \\
1 & -1 & -1 \\
1 & 0 & 1
\end{bmatrix}
$$

Therefore, the matrix representation of T is:

$$
\boxed{
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & -1 & -1 \\
1 & 0 & 1
\end{bmatrix}
}
$$

and for any vector $\mathbf{x} = (x, y, z)$:

$$
T(\mathbf{x}) = A \mathbf{x}
$$

Implications and Significance of T Being a Matrix Transformation

Practical Applications

The identification of T as a matrix transformation has significant practical repercussions. Once expressed as a matrix, it becomes straightforward to analyze, invert, and compose transformations, which is invaluable in fields like computer graphics, physics, and engineering.

Key applications include:

- Coordinate transformations: Changing from one basis or coordinate system to another.
- Data manipulation: Applying linear filters or transformations in data science.
- Modeling physical systems: Representing rotations, reflections, or scaling in three-dimensional space.
- Computer graphics: Transforming object models within 3D environments.

Further Analytical Insights

- Kernel (Null Space): Understanding the set of vectors mapped to the zero vector helps identify the transformation's injectivity.
- Range (Image): Exploring the subspace spanned by the columns of A reveals the output space.
- Determinant:

Let T R3 R3 Be Defined By T X Y Z X Y X Y Z X Z A Show That T Is A Matrix Transformation

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