Let Y Denote A Geometric Random Variable With Probability Of Success P.a Show That For A Positive Integer

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Understanding the behavior and properties of the geometric distribution is fundamental in probability theory and statistics, especially when modeling the number of trials needed to achieve the first success in a sequence of independent Bernoulli trials. In this comprehensive guide, we will explore the properties of the geometric random variable Y, which is characterized by a probability of success p, and demonstrate key results for a positive integer value of y. The discussion will include the probability mass function (pmf), cumulative distribution function (cdf), expectation, variance, and other important properties, supported by examples and derivations to enhance understanding.

Introduction to the Geometric Distribution

The geometric distribution models the number of Bernoulli trials needed to obtain the first success. It is widely used in various fields, including quality control, reliability testing, and survival analysis. The defining feature of this distribution is the independence of trials and a constant probability of success p in each trial.

Definition of the Geometric Random Variable

Let Y be a discrete random variable representing the trial count at which the first success occurs. For each trial:

- The probability of success is p, where $0 < p < 1$.
- The probability of failure is $1 - p$.

Y takes values in the set {1, 2, 3, ...}, meaning the first success can occur on the first trial, second trial, and so forth.

Assumptions

- Trials are independent.

- The probability p remains constant across trials.
- The process continues until the first success, after which the experiment stops.

Probability Mass Function (PMF) of Y

The core of understanding the geometric distribution lies in the probability mass function, which gives the probability that the first success occurs on the y-th trial.

Derivation of the PMF

For a positive integer y:

- The first y - 1 trials must be failures.
- The y-th trial must be a success.

Since trials are independent:

$$
\boxed{
P(Y = y) = (1 - p)^{y - 1} p, \quad y = 1, 2, 3, \dots
}
$$

This formula indicates that the probability decreases exponentially with y, depending on the success probability p.

Properties of the PMF

- Non-negativity: $P(Y = y) \geq 0$ for all y.
- Normalization: The sum over all possible y equals 1:

$$
\sum_{y=1}^\infty P(Y = y) = 1
$$

which can be verified by recognizing the geometric series.

Verification for a Positive Integer y

Now, we focus on demonstrating key properties for a specific positive integer y, particularly the probability $P(Y = y)$.

Step-by-Step Derivation

Given that:

$$
P(Y = y) = (1 - p)^{y - 1} p
$$

for $y \geq 1$, the derivation is straightforward:

1. Probability of failure in first y - 1 trials:

$$
P(\text{fail in trial 1}) \times P(\text{fail in trial 2}) \times \dots \times P(\text{fail in trial } y-1) = (1 - p)^{y - 1}
$$

2. Probability of success in the y-th trial:

$$
p
$$

3. Combined probability:

$$
P(Y = y) = (1 - p)^{y - 1} \times p
$$

This confirms the formula for the pmf.

Implication of the Formula

- The probability decreases exponentially as y increases.
- For larger y, the chance that the first success occurs at that trial diminishes.
- The distribution is skewed towards smaller y when p is high.

Expected Value and Variance of the Geometric Distribution

Understanding the mean and variance of Y provides insight into the typical number of trials needed for success and the variability around this average.

Expected Value (Mean)

The expected value of Y, denoted $E[Y]$, is obtained by summing over all possible values:

$$
E[Y] = \sum_{y=1}^\infty y \cdot P(Y = y)
$$

Using the pmf:

$$
E[Y] = \sum_{y=1}^\infty y (1 - p)^{y - 1} p
$$

This is a well-known sum in probability theory, which evaluates to:

$$
\boxed{
E[Y] = \frac{1}{p}
}
$$

Intuitive Explanation:

- If p is high (close to 1), the expected number of trials is close to 1.
- If p is low, more trials are expected before success.

Variance of Y

Variance measures the spread of the distribution:

$$
\operatorname{Var}(Y) = E[Y^2] - (E[Y])^2
$$

Calculating $E[Y^2]$:

$$
E[Y^2] = \sum_{y=1}^\infty y^2 P(Y = y)

$$

This sum evaluates to:

$$
E[Y^2] = \frac{2 - p}{p^2}
$$

Thus, the variance is:

$$
\boxed{
\operatorname{Var}(Y) = \frac{1 - p}{p^2}
}
$$

Summary:

Parameter	Formula
Expected value $E[Y]$	$\frac{1}{p}$
Variance $\operatorname{Var}(Y)$	$\frac{1 - p}{p^2}$

Distributional Properties and Related Results

Besides the pmf, the geometric distribution exhibits several other significant properties.

Cumulative Distribution Function (CDF)

The cdf $F(y) = P(Y \leq y)$ for $y \geq 1$ is:

$$
F(y) = 1 - (1 - p)^y
$$

This represents the probability that the first success occurs on or before trial y.

Memoryless Property

One of the defining features of the geometric distribution is its memoryless property:

$$
P(Y > s + t \mid Y > s) = P(Y > t)
$$

for all $s, t \geq 0$. This indicates that the process "restarts" after any number of failures, which has important implications in modeling and analysis.

Relation to Other Distributions

- The geometric distribution is related to the negative binomial distribution, which counts the number of trials up to a fixed number of successes.
- When considering the number of failures before the first success, the distribution is often called the geometric distribution in the "failures before success" formulation.

Applications and Examples

Understanding the properties of Y helps in practical scenarios.

Example 1: Quality Control

Suppose a factory produces items with a defect probability p = 0.05. The number of items checked until the first defective item appears is modeled by Y.

- The probability that the first defective appears on the 10th item:

$$
P(Y=10) = (1 - 0.05)^{9} \times 0.05 \approx 0.95^{9} \times 0.05
$$

- The expected number of items checked before the first defect:

$$
E[Y] = \frac{1}{0.05} = 20
$$

- Variability around this mean:

$$
\operatorname{Var}(Y) = \frac{1 - 0.05}{(0.05)^2} = \frac{0.95}{0.0025} = 380
$$

\]

Example 2: Reliability Testing

In a reliability test, the number of cycles until failure follows a geometric distribution with a success probability p of failure per cycle. For p = 0.01:

- The probability that the system survives past 100 cycles:

\[
P(Y > 100) = (1 - p)^{100} = 0.99^{100} \approx 0.366
\]

- The expected number of cycles before failure:

\[
E[Y] = 100
\]

Conclusion and Summary

The geometric distribution provides a powerful model for scenarios involving independent Bernoulli trials until the first success. For a positive integer y, the probability that the first success occurs exactly on trial y is given by:

\[
P(Y = y) = (1 - p)^{y - 1} p
\]

This fundamental formula encapsulates the distribution's exponential decay and forms the basis for deriving other properties such as the mean, variance, and cumulative probabilities.

Key takeaways:

- The distribution is discrete, with support on the positive integers.
- The expected number of trials until the first success is \(1/p\).
- Variability in

Frequently Asked Questions

What is a geometric random variable and how is it defined with respect to probability of success p?

A geometric random variable Y represents the number of trials needed to achieve the first success in a sequence of independent Bernoulli trials, each with success probability p.

How do you calculate the probability that the geometric random variable Y equals a specific positive integer k?

The probability that Y equals k is given by $P(Y = k) = (1 - p)^{k - 1} p$, for $k = 1, 2, 3, ...$

What is the expected value (mean) of a geometric random variable Y with success probability p?

The expected value $E[Y]$ is $1 / p$.

How can we derive the variance of a geometric random variable Y with parameter p?

The variance $Var(Y)$ is $(1 - p) / p^2$.

Show that the sum of independent geometric random variables with the same success probability p follows a negative binomial distribution.

Yes, the sum of n independent geometric variables with success probability p has a negative binomial distribution with parameters n and p, representing the number of trials needed to achieve n successes.

What is the cumulative distribution function (CDF) of a geometric random variable Y?

The CDF is $F(k) = P(Y \leq k) = 1 - (1 - p)^k$, for $k = 1, 2, 3, ...$

How does the geometric distribution relate to the negative binomial distribution for a fixed number of successes?

The geometric distribution is a special case of the negative binomial distribution when the number of successes $n = 1$.

Show the derivation of the probability that the first success occurs on the k-th trial for the geometric variable Y.

This probability is derived as P(Y = k) = (1 - p)^{k - 1} p, since the first k - 1 trials are failures and the k-th is a success.

Is the geometric distribution memoryless? Explain with respect to P(Y > s + t | Y > s).

Yes, the geometric distribution is memoryless, meaning P(Y > s + t | Y > s) = P(Y > t).

How can the properties of a geometric random variable be used in modeling real-world scenarios?

They are useful in modeling scenarios like the number of attempts until a success in quality control, network packet transmissions, or success in games of chance, leveraging its memoryless property and simple probability structure.

Additional Resources

Let Y Denote A Geometric Random Variable With Probability Of Success P. Show That For A Positive Integer

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Introduction

The geometric distribution is one of the foundational probability distributions in statistical theory, often encountered in scenarios involving discrete, Bernoulli trials. Its simplicity and intuitive interpretation make it a staple in both theoretical investigations and practical applications such as quality control, reliability testing, and modeling of waiting times.

At the heart of the geometric distribution lies the concept of counting the number of trials until the first success occurs, where each trial is independent and has a fixed probability of success, denoted by \(p \). This article aims to delve deeply into the properties of the geometric random variable \(Y \), especially focusing on its probability mass function (pmf), moments, and the derivation of key results for positive integers.

The goal is to rigorously demonstrate, for a positive integer \(k \), that the probability \(P(Y = k) \) adheres to the well-known form:

\[

$$
P(Y = k) = (1 - p)^{k-1} p, \quad \text{for } k = 1, 2, 3, \ldots
$$

This proof forms the backbone for understanding the distribution's characteristics and provides a foundation for more advanced statistical modeling.

Background and Definition of the Geometric Distribution

The Concept of a Geometric Random Variable

A geometric random variable Y models the number of Bernoulli trials needed to achieve the first success, where each trial is independent and has the same probability p of success.

- Key assumptions:
- Independence of trials.
- Constant probability of success p in each trial.
- The process continues until the first success occurs.

Formal Definition

Let Y be a discrete random variable representing the trial count at which the first success occurs. Then:

$$
Y = \text{the trial number of the first success}
$$

The probability mass function (pmf) of Y is given by:

$$
P(Y = k) = ?
$$

where $k \in \{1, 2, 3, \ldots\}$.

The task is to derive this pmf explicitly and verify that it takes the form:

$$
P(Y = k) = (1 - p)^{k-1} p
$$

Derivation of the Probability Mass Function

Intuitive Approach

The event $Y = k$ signifies that:

- The first $k-1$ trials are failures.
- The k^{th} trial is a success.

Given the independence of trials and the fixed success probability, the probability of this sequence is:

$$
(\text{fail})^{k-1} \times (\text{success}) = (1 - p)^{k-1} p
$$

since each failure occurs with probability $1 - p$.

Formal Proof

Step 1: Define the event

$$
\{Y = k\} = \{\text{fail in trials } 1, 2, ..., k-1\} \cap \{\text{success in trial } k\}
$$

Step 2: Use independence

Because each trial is independent:

$$
P(Y = k) = P(\text{fail in first } k-1 \text{ trials}) \times P(\text{success in } k^{th} \text{ trial})
$$

which simplifies to:

$$
P(Y = k) = (1 - p)^{k-1} \times p
$$

Step 3: Verify the pmf

To confirm that this is a valid pmf, sum over all possible k:

$$
\sum_{k=1}^{\infty} P(Y = k) = \sum_{k=1}^{\infty} (1 - p)^{k-1} p
$$

Recognize this as a geometric series:

$$
p \sum_{k=1}^{\infty} (1 - p)^{k-1} = p \times \frac{1}{1 - (1 - p)} = p \times \frac{1}{p} = 1
$$

\]

which confirms that the pmf sums to 1, satisfying the axioms of probability.

Properties of the Geometric Distribution

Expectation and Variance

The geometric distribution's moments are fundamental for understanding its behavior.

Expected Value

$$
E[Y] = \frac{1}{p}
$$

This indicates that the expected number of trials until the first success increases as p decreases.

Variance

$$
Var(Y) = \frac{1 - p}{p^2}
$$

which quantifies the variability around the mean.

Memoryless Property

A key characteristic of the geometric distribution is its memoryless property:

$$
P(Y > s + t \mid Y > s) = P(Y > t)
$$

for all $s, t \geq 0$. This means that the waiting time until the first success has no memory of how long we've already waited.

Generalization to the Negative Binomial Distribution

While the geometric distribution models the number of trials until the first success, its generalization, the negative binomial distribution, models the number of trials until r successes. The pmf for the negative binomial (with r successes) is:

$$
P(Y = k) = \binom{k-1}{r-1} p^{r} (1 - p)^{k - r}
$$

for $k \geq r$.

This broader context underscores the significance of the geometric distribution as a special case with $r = 1$.

Application and Implications

Practical Scenarios

- Quality Control: Determining the number of items inspected until a defective item is found.
- Reliability Engineering: Estimating the number of tests until a component fails.
- Biology: Modeling the number of trials until a mutant appears.

Limitations

- Assumes independent trials with constant success probability.
- Not suitable if the probability of success varies over trials.

Extensions

- The geometric distribution can be adapted for different contexts, such as the shifted geometric, where the count starts from zero.
- The negative binomial extends its utility to count the number of trials needed for multiple successes.

Conclusion

The geometric distribution, characterized by the pmf $P(Y = k) = (1 - p)^{k-1} p$, provides a simple yet powerful model for waiting times in Bernoulli processes. Its derivation hinges on the independence of trials and the fixed probability of success, leading to an elegant geometric series summation that confirms the pmf's validity.

Understanding this distribution's fundamental properties, such as expectation, variance, and memoryless nature, equips statisticians and researchers with essential tools for modeling real-world phenomena involving discrete waiting times. The rigorous proof for the pmf for positive integers k consolidates its role as a cornerstone of discrete probability theory, with broad applicability across scientific disciplines.

References

- Ross, S. M. (2014). Introduction to Probability Models. Academic Press.
- Johnson, N. L., Kotz, S., & Balakrishnan, N. (1994). Continuous Univariate Distributions. Wiley-Interscience.
- Feller, W. (1968). An Introduction to Probability Theory and Its Applications. Wiley.

Note: This article provides a comprehensive analysis suitable for academic review and understanding of the geometric distribution's core properties and derivation.

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