Prove (f_n) Does Not Converge Uniformly Using Epsilon Criteria: For Any Natural Number N, For All N >=

Prove (f_n) Does Not Converge Uniformly Using Epsilon Criteria: For Any Natural Number N, For All N ≥

Understanding whether a sequence of functions converges uniformly is a fundamental aspect of analysis, especially in the context of functional sequences. In this article, we will delve into a rigorous proof demonstrating that a given sequence (f_n) does not converge uniformly using the epsilon (ε) criteria. The core idea hinges on showing that, for any natural number N, there exists a point in the domain where the difference between the functions exceeds a fixed ε, thus violating the condition for uniform convergence.

Preliminaries: Definitions and Concepts

Before embarking on the proof, it's essential to clarify the key concepts involved.

Pointwise vs. Uniform Convergence

- **Pointwise Convergence:** A sequence of functions (f_n) converges pointwise to a function f on a domain D if, for every $x \in D$, $\lim_{n \to \infty} f_n(x) = f(x)$.

- **Uniform Convergence:** (f_n) converges uniformly to f on D if, for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in D$, $|f_n(x) - f(x)| < \varepsilon$.

The Epsilon Criterion for Uniform Convergence

- The formal statement: $\lim_{n \to \infty} f_n = f$ uniformly if and only if:
$$
\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \forall x \in D, |f_n(x) - f(x)| < \varepsilon
$$
- To disprove uniform convergence, it suffices to show that:
$$
\exists \varepsilon_0 > 0 \text{ such that } \forall N \in \mathbb{N}, \exists n \geq N,

$$
\text{ and } x \in D \text{ with } |f_n(x) - f(x)| \geq \varepsilon_0
$$

The Sequence (f_n) Under Consideration

Suppose we analyze the sequence of functions:
$$
f_n(x) = x^n
$$
defined on the domain $D = [0, 1]$.

Intuition:
- For each fixed $x \in [0, 1)$, as $n \to \infty$, $x^n \to 0$.
- For $x = 1$, $f_n(1) = 1^n = 1$ for all n.

Pointwise limit function:
$$
f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases}
0, & 0 \leq x < 1 \\
1, & x = 1
\end{cases}
$$

The question is: does (f_n) converge uniformly to f on $[0, 1]$?

Key Idea for the Proof

To prove that (f_n) does not converge uniformly to f, we need to demonstrate that:

- There exists an $\varepsilon_0 > 0$ such that, no matter how large N is, we can find an $n \geq N$ and a point $x \in [0, 1]$ with $|f_n(x) - f(x)| \geq \varepsilon_0$.

In this case, aiming for $\varepsilon_0 = \frac{1}{2}$ (or any positive number less than 1), we will show that the supremum difference between f_n and f remains above ε_0, regardless of how large n is.

Step-by-Step Proof: (f_n) Does Not Converge

Uniformly

Step 1: Identify the Candidate ε_0

- Choose $\varepsilon_0 = \frac{1}{2}$. Our goal is to show:
$$
\forall N \in \mathbb{N}, \exists n \geq N, \exists x \in [0, 1], |f_n(x) - f(x)| \geq \frac{1}{2}
$$

Step 2: Analyze the Behavior at $x = 1$

- For all n,
$$
f_n(1) = 1
$$
- The limit function at $x=1$ is $f(1) = 1$, so the difference:
$$
|f_n(1) - f(1)| = 0
$$
- Not helpful in demonstrating non-uniform convergence since the difference vanishes at $x=1$.

Step 3: Focus on x close to 1 but less than 1

- The worst-case difference occurs when $f(x) = 0$ but $f_n(x)$ is not close to 0.
- For $x \in [0, 1)$, $f(x) = 0$, so the difference simplifies to:
$$
|f_n(x) - 0| = |x^n| = x^n
$$

Step 4: Finding the Point x_n for a Given n

- To guarantee $|x^n| \geq \frac{1}{2}$, solve:
$$
x^n \geq \frac{1}{2}
$$
- Equivalently:
$$
x \geq \left(\frac{1}{2}\right)^{1/n}
$$

- For each n, pick:
$$
x_n = \left(\frac{1}{2}\right)^{1/n}
$$
which is in $[0, 1)$.

Step 5: Verifying the Difference at x_n

- At $x = x_n$,
$$
|f_n(x_n) - f(x_n)| = x_n^n = \left(\frac{1}{2}\right)^{1/n \cdot n} = \frac{1}{2}
$$
- Since the difference is exactly $\frac{1}{2}$, which equals ε_0, this point demonstrates the failure of the uniform convergence condition.

Step 6: Final Contradiction with the Epsilon Criterion

- For any $N \in \mathbb{N}$, choose $n \geq N$.
- The corresponding $x_n = (1/2)^{1/n}$ satisfies:
$$
|f_n(x_n) - f(x_n)| = \frac{1}{2} \geq \varepsilon_0
$$
- This shows that, no matter how large N is, the supremum difference:
$$
\sup_{x \in [0, 1]} |f_n(x) - f(x)| \geq \frac{1}{2}
$$
- Since the supremum does not tend to zero as $n \to \infty$, the sequence (f_n) does not converge uniformly to f.

Conclusion: The Sequence (f_n) Does Not Converge Uniformly

The detailed analysis reveals that the sequence of functions $f_n(x) = x^n$ on $[0, 1]$ fails to satisfy the epsilon criterion for uniform convergence. Specifically, by selecting $\varepsilon_0 = 1/2$, we have demonstrated:

- For any natural number N, there exists an $n \geq N$ and a point $x_n = (1/2)^{1/n}$ such that:
$$
|f_n(x_n) - f(x_n)| = \frac{1}{2}
$$
- Therefore, the supremum of $|f_n(x) - f(x)|$ over $[0, 1]$ stays bounded below by $1/2$, contradicting the requirement that it should tend to zero for uniform convergence.

Hence, the sequence (f_n) does not converge uniformly to its pointwise limit f on $[0,1]$.

Additional Insights and Generalizations

This proof technique can be adapted for other sequences of functions by:

- Identifying points where the functions deviate significantly from the limit function

Frequently Asked Questions

What is the main idea behind proving that a sequence of functions (f_n) does not converge uniformly using the epsilon criterion?

The main idea is to show that there exists some epsilon > 0 such that, for any natural number N, there is some n ≥ N and a point x in the domain where |f_n(x) - f(x)| ≥ epsilon, thereby violating the uniform convergence condition.

How does the epsilon criterion define uniform convergence of a sequence of functions (f_n) to a function f?

A sequence (f_n) converges uniformly to f if, for every epsilon > 0, there exists N such that for all n ≥ N and all x in the domain, |f_n(x) - f(x)| < epsilon.

When trying to prove that (f_n) does not converge uniformly, what role does the statement 'for any N, there exists n ≥ N such that...' play?

This statement emphasizes that no matter how large N is chosen, one can find some n ≥ N where the convergence condition fails at some point x, demonstrating that uniform convergence does not occur.

Can you give an example of a sequence (f_n) that does not converge uniformly, and how to use epsilon criteria to prove it?

Consider f_n(x) = x^n on [0,1]. To prove it does not converge uniformly to the zero function, pick epsilon = 0.5. For any N, choose n ≥ N and x close to 1, say x = 1 - 1/n, then |f_n(x) - 0| = (1 - 1/n)^n ≈ e^{-1} > 0.5, violating the uniform convergence condition.

Why is it important to consider 'for any N' when using

epsilon criteria to show non-uniform convergence?

Because uniform convergence requires that beyond some N, all subsequent functions are close to the limit uniformly over the entire domain. Showing that for every N, there exists $n \geq N$ where the convergence fails demonstrates that this condition cannot be met.

How does the choice of epsilon influence the proof that (f_n) does not converge uniformly?

Selecting a specific epsilon (usually a positive value) helps identify points and indices where the difference $|f_n(x) - f(x)|$ exceeds epsilon, which is essential for demonstrating the failure of uniform convergence.

What is the significance of the phrase 'for all $N \geq 1$' in the context of proving non-uniform convergence?

It indicates that no matter how large N is chosen, the sequence still fails to satisfy the uniform convergence condition, reinforcing that the convergence is not uniform.

How does the epsilon criterion relate to the concept of pointwise vs. uniform convergence?

While pointwise convergence only requires that for each fixed x, the sequence $(f_n(x))$ converges to $f(x)$, uniform convergence demands that the convergence happens uniformly over the entire domain. Using epsilon criteria helps distinguish between these by testing the uniformity of the convergence.

What is a common strategy to prove a sequence (f_n) does not converge uniformly using the epsilon approach?

A common strategy is to fix an epsilon > 0 and then show that for every N, you can find an $n \geq N$ and a point x such that $|f_n(x) - f(x)| \geq$ epsilon, thus violating the requirement for uniform convergence.

Additional Resources

Prove (f_n) Does Not Converge Uniformly Using Epsilon Criteria: For Any Natural Number N, For All N ≥

Understanding the convergence of sequences and functions is a fundamental aspect of real analysis, especially when it comes to uniform convergence. When dealing with a sequence of functions (f_n), establishing whether or not it converges uniformly involves rigorous criteria, primarily centered around epsilon (ε) definitions. This article explores the methodology to prove that a sequence (f_n) does not converge uniformly, emphasizing the epsilon criteria, and demonstrating that for any given natural number N, there exists an $n \geq N$ such that the functions f_n do not satisfy uniform convergence conditions.

Understanding Uniform Convergence and Epsilon Criteria

Before diving into the proof techniques, it is essential to grasp what uniform convergence entails and how epsilon criteria serve as the backbone of such proofs.

What Is Uniform Convergence?

A sequence of functions (f_n) defined on a set $D \subseteq \mathbb{R}$ converges uniformly to a function f if:

$$
\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \forall x \in D, |f_n(x) - f(x)| < \varepsilon
$$

This definition emphasizes that the convergence does not depend on the point x; the same N works uniformly for all $x \in D$.

The Role of Epsilon (ε) in Convergence

The epsilon criterion is a quantitative measure of how close f_n gets to f. To show that (f_n) converges uniformly, we need to find a threshold N such that beyond this, all f_n are within ε of f uniformly over the domain.

Conversely, to prove that (f_n) does not converge uniformly, we typically demonstrate that:

$$
\exists \varepsilon_0 > 0, \text{ such that } \forall N \in \mathbb{N}, \exists n \geq N, \text{ and } x \in D, |f_n(x) - f(x)| \geq \varepsilon_0
$$

In essence, no matter how large N is chosen, there will always be some $n \geq N$ and some point $x \in D$ for which the functions are not sufficiently close.

Approach to Proving Non-Uniform Convergence

The general strategy involves constructing explicit counterexamples or showing the existence of points where the difference exceeds a given ε. The key steps include:

1. Identify the candidate limit function f.
2. Choose a suitable $\varepsilon > 0$.
3. For any $N \in \mathbb{N}$, find an $n \geq N$ and a point x such that $|f_n(x) - f(x)| \geq \varepsilon$.

This process often involves selecting points where the sequence behaves "worst" and demonstrating that the uniform bound cannot be achieved.

Detailed Example: Sequence (f_n) and Its Non-Convergence

Let's examine a classic example to illustrate the proof technique:

Sequence Definition:

$$
f_n(x) = x^n \quad \text{for } x \in [0, 1]
$$

Expected Limit:

$$
f(x) = \begin{cases}
0, & 0 \leq x < 1 \\
1, & x=1
\end{cases}
$$

In this case, (f_n) converges pointwise to f, but not uniformly on $[0, 1]$.

Proving Non-Uniform Convergence

Step 1: Choose $\varepsilon = \frac{1}{2}$.

Step 2: For any $N \in \mathbb{N}$, find $n \geq N$ and $x \in [0, 1]$ such that:

$$
|f_n(x) - f(x)| \geq \frac{1}{2}
$$

Step 3: Selection of x for the proof

Since $f(x) = 0$ for $x < 1$, and $f(1) = 1$, consider $x = (1 - \delta)$ for small $\delta > 0$. For $x \in [0, 1)$:

$$
f_n(x) = x^n = (1 - \delta)^n
$$

As $n \to \infty$:

$$
(1 - \delta)^n \to 0
$$

but for finite n, we can control the size by choosing δ.

Step 4: Show that for any N, there exists $n \geq N$ with

$$
|f_n(1) - f(1)| = |1 - 1| = 0
$$

which is trivial; the problematic point is approaching $x=1$ from below.

Alternatively, to demonstrate non-uniform convergence, look at $x = 1 - \frac{1}{n}$:

$$
f_n\left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^n
$$

which tends to $e^{-1} \approx 0.3679$, while the limit at this point is 0. So:

$$
|f_n(1 - \frac{1}{n}) - f(1 - \frac{1}{n})| = \left| \left(1 - \frac{1}{n}\right)^n - 0 \right| \approx 0.368
$$

which is greater than $\varepsilon = 0.5$ for sufficiently large n, specifically when:

$$
\left(1 - \frac{1}{n}\right)^n \geq 0.5
$$

This inequality holds for all $n \leq 3$. For larger n, the difference becomes less than 0.5, but the key insight is that no matter how large N is, choosing n small enough (say, $n = 2$) yields:

$$
|f_2(1 - 1/2) - 0| = (1 - 1/2)^2 = (1/2)^2 = 1/4 < 0.5
$$

which is less than ε.

Thus, the general proof involves selecting points approaching 1 from below and showing that the maximum deviation does not go uniformly to zero.

General Theorem for Non-Uniform Convergence

A formal statement:

Theorem:
Let (f_n) be a sequence of functions defined on a domain D, with pointwise limit f. If:

$$
\sup_{x \in D} |f_n(x) - f(x)| \not\to 0 \quad \text{as } n \to \infty
$$

then (f_n) does not converge uniformly to f.

Proof Sketch:
Given the above, for some $\varepsilon_0 > 0$, there exists an infinite subsequence (n_k) such that:

$$
\sup_{x \in D} |f_{n_k}(x) - f(x)| \geq \varepsilon_0
$$

This directly implies that the epsilon criterion for uniform convergence fails because, for any N, we can find $n \geq N$ with:

$$
\sup_{x \in D} |f_n(x) - f(x)| \geq \varepsilon_0
$$

since otherwise, the sequence would converge uniformly to f.

Features and Pros/Cons of the Proof Technique

Features:

- Constructive approach: Often involves explicitly identifying points where the difference exceeds ε.
- General applicability: Can be used for sequences of functions with complex behaviors.
- Intuitive understanding: Highlights the failure of the "uniform" aspect of convergence.

Pros:

- Provides concrete counterexamples demonstrating non-uniform convergence.
- Clarifies how local deviations prevent uniform convergence.
- Useful for educational purposes to illustrate the difference between pointwise and uniform convergence.

Cons:

- May require intricate analysis or clever point selection.
- Not always straightforward for complex function sequences.
- Sometimes relies on understanding subtle behaviors near domain boundaries.

Conclusion

Proving that a sequence (f_n) does not converge uniformly often hinges on

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