Show That There Is No Acceptable Solution To The (time-independent) Schrodinger Equation For The Infinite

Show That There Is No Acceptable Solution To The (Time-Independent) Schrödinger Equation For The Infinite Potential Well

Introduction

The Schrödinger equation forms the cornerstone of non-relativistic quantum mechanics, describing how the quantum state of a physical system evolves or is structured. When considering potential energy landscapes, certain idealized models—such as the infinite potential well—are frequently employed to elucidate fundamental concepts. However, these models often involve idealizations that lead to mathematical and physical inconsistencies. This article aims to demonstrate that there is no physically acceptable solution to the time-independent Schrödinger equation for the infinite potential well, emphasizing the mathematical and conceptual issues that arise in the limiting case of an infinitely high potential barrier.

Understanding the Infinite Potential Well Model

The infinite potential well, also known as the infinite square well, is a simplified model where a particle is confined within a region of space by infinitely high potential barriers at the boundaries. Formally, the potential $V(x)$ is described as:

$$
V(x) =
\begin{cases}
0, & 0 < x < a \\
\infty, & \text{elsewhere}
\end{cases}
$$

The main features of this model include:
- The particle cannot exist outside the interval $[0, a]$.
- The wavefunction $\psi(x)$ must vanish at the boundaries $x=0$ and $x=a$, because the probability of finding the particle outside the well is zero and the potential is infinitely high there.

The time-independent Schrödinger equation within the well (where $V(x)=0$) reduces to:

$$
-\frac{\hbar^2}{2m}\frac{d^2 \psi(x)}{dx^2} = E \psi(x)
$$

with boundary conditions:

$$
\psi(0) = 0, \quad \psi(a) = 0
$$

The solutions are well-known for finite potential wells; however, when considering the limit as the potential barriers tend to infinity, some fundamental issues emerge.

Mathematical Foundations and Boundary Conditions

The Schrödinger Equation and Its Solutions in Finite Wells

For a finite potential well, the eigenfunctions inside the well are sinusoidal:

$$
\psi_n(x) = \sqrt{\frac{2}{a}} \sin \left(\frac{n \pi x}{a} \right), \quad n=1,2,3,\dots
$$

with corresponding energies:

$$
E_n = \frac{\hbar^2 \pi^2 n^2}{2 m a^2}
$$

These wavefunctions satisfy the boundary conditions $\psi(0) = 0$ and $\psi(a) = 0$. Outside the well, the wavefunctions decay exponentially if the potential is finite, and the solutions are physically acceptable as long as the energy is less than the potential barrier height.

The Limit of Infinite Potential: Mathematical Implications

When the potential barriers become infinitely high ($V_0 \to \infty$), the wavefunctions outside the well must become zero to satisfy the boundary conditions, and the solutions inside the well are confined strictly within $[0, a]$. Despite this, as the potential tends to infinity, the formal solutions inside the well are unaffected in form, but the behavior outside the well becomes problematic:

- The wavefunction must be zero outside the well, which is consistent with the boundary conditions.
- The wavefunction inside remains sinusoidal, with normalization constants ensuring total probability equals one.

However, this idealization introduces issues:
- The potential is no longer a well-behaved finite function; it becomes a discontinuous jump to infinity.
- The limit process involves taking a sequence of finite potentials $V_0 \to \infty$, but this sequence does not necessarily produce a well-defined solution in the limit.

Physical and Mathematical Problems with the Infinite Potential Well

Non-Existence of a Well-Defined Limit Solution

One core issue is that the solutions for finite wells depend on the finite potential height V_0. As $V_0 \to \infty$:

- The eigenvalues E_n tend to finite limits, but their derivation relies on the potential being finite.
- The wavefunctions outside the well tend to zero exponentially, but outside the well, the wavefunction is strictly zero in the infinite limit.
- The process of taking the limit $V_0 \to \infty$ does not produce a convergent sequence of functions in the usual sense, because the boundary conditions change discontinuously.

In particular:
- The wavefunctions become discontinuous at the boundaries if one attempts to define them directly at the limit.
- The boundary conditions become "hard" constraints, which cannot be derived as limits of finite well solutions, but are imposed ad hoc.

Mathematical Inconsistencies and Distributional Solutions

Mathematically, the wavefunction in the infinite well is better described as a distribution rather than a classical function. This leads to:
- The wavefunction being zero outside the well and a sinusoid inside.
- The derivative of the wavefunction being discontinuous at the boundaries, which can be problematic in the context of the differential equation.

This discontinuity violates the assumptions underlying the differential equation and indicates that the solution is not a classical function but a generalized function (distribution).

Physical Implausibility of Infinite Potentials

From a physical standpoint, infinitely high potential barriers are unphysical:
- No real potential can be truly infinite.
- The concept of an infinitely abrupt boundary is an idealization that cannot be realized physically.
- The wavefunction's behavior at such boundaries involves infinite derivatives, which are non-physical and cannot be realized in actual systems.

Thus, the solutions derived for the infinite well do not correspond to any physically realizable state, reinforcing the argument that no physically acceptable solutions exist for the idealized case.

Conclusion: Why There Is No Acceptable Solution

Summary of Mathematical and Physical Arguments

- The solutions for finite potential wells depend on the potential height \(V_0\), and as this tends to infinity, the solutions do not converge in a classical sense.
- The boundary conditions in the infinite potential well are imposed ad hoc, rather than derived as the limit of finite potentials.
- The discontinuities in the wavefunction's derivatives at the boundaries violate the smoothness required by the Schrödinger differential equation.
- The idealization involves an unphysical potential, leading to solutions that are not physically realizable.

Implications for Quantum Mechanics

The infinite potential well serves as an instructive mathematical model, but it should be understood as an approximation or idealization rather than a physically exact scenario. The non-existence of acceptable solutions in the strict limit underscores:
- The importance of considering finite potentials in realistic models.
- The limitations of idealized boundary conditions and the necessity of physically plausible potentials.

Final Remarks

In conclusion, the argument that there is no acceptable solution to the Schrödinger equation for the infinite potential well hinges on both mathematical inconsistencies and physical considerations. The solutions are not limits of physically meaningful solutions for finite potentials, and the boundary conditions imposed are not derivable from the differential equation itself in the limit. Therefore, the infinite potential well is best regarded as an idealized approximation, and one must be cautious in interpreting its solutions as physically exact. This insight emphasizes the importance of considering physical realism in quantum mechanical models and understanding the limitations of idealizations such as infinitely high potential barriers.

Frequently Asked Questions

Why does the time-independent Schrödinger equation have no acceptable solutions for the infinite potential well?

Because the infinite potential well imposes boundary conditions where the wavefunction must be zero at the boundaries, leading to solutions that are only physically acceptable if the wavefunction is zero everywhere, which is trivial and not a valid state.

What is the fundamental reason that no non-trivial solutions exist for the Schrödinger equation with an infinite potential barrier?

The infinite potential barrier causes the wavefunction to vanish at the boundary, but the differential equation's boundary conditions only admit solutions that are identically zero, implying no physical states can exist within the infinite well.

How does the boundary condition at infinity affect the solutions of the Schrödinger equation?

At infinity, the wavefunction must tend to zero for bound states; however, in the case of an infinite potential, the boundary conditions are so restrictive that they prevent any non-trivial solutions from existing.

Can the Schrödinger equation be solved for an infinitely high potential, and what is the issue?

While formally the equation can be written down, the boundary conditions for an infinite potential well lead to solutions that are identically zero, indicating that acceptable non-trivial solutions do not exist.

Why are the eigenfunctions of the infinite potential well considered physically acceptable solutions, despite the mathematical issues?

Because they satisfy the boundary conditions of vanishing at the walls and are normalizable, but mathematically, the idealized infinite potential creates boundary conditions that eliminate all non-trivial solutions, highlighting the idealization's limitations.

Is the non-existence of solutions to the Schrödinger

equation for the infinite potential a mathematical or physical problem?

It is primarily a mathematical issue arising from the idealized boundary conditions; physically, infinite potentials are approximations, and real systems have finite barriers allowing solutions.

What role does the concept of self-adjointness play in the non-existence of solutions for the infinite potential case?

Self-adjointness of the Hamiltonian operator ensures real eigenvalues and physical solutions; in the case of an infinite potential, the boundary conditions disrupt self-adjointness, leading to the absence of acceptable solutions.

How does the concept of limiting processes relate to the solution of the Schrödinger equation with infinite potential?

The infinite potential can be viewed as a limit of large finite potentials; in this limit, solutions tend to zero within the well, indicating that true infinite potential solutions are non-physical and do not exist in the strict mathematical sense.

What is the significance of the statement 'no acceptable solution' in the context of the infinite potential well problem?

It signifies that, under strict boundary conditions of an idealized infinite potential, the Schrödinger equation admits only the trivial solution, implying no physically meaningful bound states exist in this idealization.

Additional Resources

Show That There Is No Acceptable Solution To The (time-independent) Schrödinger Equation For The Infinite

In the realm of quantum mechanics, the Schrödinger equation stands as a fundamental pillar, describing how quantum states evolve and how particles behave at microscopic scales. Among its various forms, the time-independent Schrödinger equation is particularly instrumental in understanding stationary states within potential wells and barriers. However, when it comes to the idealized concept of an infinite potential well, a classical question arises: Is there an acceptable solution to the Schrödinger equation that satisfies all the physical and mathematical criteria? The answer, as we shall explore, is nuanced and reveals deep insights into the nature of quantum states and the limits of idealized models.

Understanding the Infinite Potential Well

Before delving into the core proof or argument, it's essential to understand what the infinite potential well entails.

The Model

- Definition: The infinite potential well, also known as the particle in a box, is a theoretical construct where a particle is confined within a region of space, say between $x=0$ and $x=a$, with the potential energy $V(x)$ defined as:

$$
V(x) = \begin{cases}
0, & 0 < x < a \\
\infty, & \text{elsewhere}
\end{cases}
$$

- Physical implication: The particle cannot exist outside the boundaries because the potential energy there is infinitely large, effectively trapping it within $0 < x < a$.

The Schrödinger Equation in This Context

The time-independent Schrödinger equation inside the well simplifies to:

$$
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} = E \psi(x)
$$

where $\hbar$ is the reduced Planck's constant, m is the particle's mass, E is the energy eigenvalue, and $\psi(x)$ is the wavefunction.

The Boundary Conditions and Their Consequences

Since the potential is infinite outside the well, the wavefunction must vanish at the boundaries:

$$
\psi(0) = 0, \quad \psi(a) = 0
$$

This ensures the particle's probability of being outside the well is zero, consistent with the infinite potential barrier.

Now, the core question arises:

> Can the Schrödinger equation admit solutions that satisfy these boundary conditions

and are physically acceptable?

The Formal Solution Inside the Well

Within the well ($0 < x < a$), the general solution to the differential equation is:

$$
\psi(x) = A \sin(kx) + B \cos(kx)
$$

where

$$
k = \sqrt{\frac{2mE}{\hbar^2}}
$$

Given the boundary conditions:

- At $x=0$:

$$
\psi(0) = B = 0
$$

- At $x=a$:

$$
\psi(a) = A \sin(ka) = 0
$$

For a non-trivial solution ($A \neq 0$), the sine term must vanish at $x=a$:

$$
\sin(ka) = 0 \Rightarrow ka = n\pi, \quad n=1,2,3,\dots
$$

Thus,

$$
k = \frac{n\pi}{a}
$$

and the corresponding energy eigenvalues are:

$$
E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2 n^2}{2m a^2}
$$

The normalized eigenfunctions are:

$$
\psi_n(x) = \sqrt{\frac{2}{a}} \sin \left(\frac{n\pi x}{a} \right)
$$

The Crux: Infinite Potential and the Physical Acceptability of Solutions

While mathematically these solutions seem straightforward, the core issue lies in the idealization of the potential as infinite. How does this affect the physical acceptability of these solutions?

The Mathematical Idealization vs. Physical Reality

- The infinite potential well is a limit case—a mathematical abstraction—since real physical barriers are finite, albeit possibly very high.
- In the idealized case, the wavefunction is strictly zero outside the well, which simplifies calculations but introduces issues regarding the nature of the solutions at the boundaries.

The Boundary at Infinity: Is the Solution Truly Acceptable?

One might argue that because the wavefunctions vanish at the boundary and are finite everywhere inside, the solutions are physically acceptable. However, certain subtle points merit deeper analysis:

1. Discontinuity in the Derivative at the Boundary:

Since the potential jumps abruptly from zero to infinity, the wavefunction's derivatives must satisfy the Schrödinger equation. The second derivative of ψ relates to the potential:

$$
-\frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi
$$

But at the boundary, the potential is infinite, which implies that the second derivative of the wavefunction must also become unbounded or undefined to satisfy the differential equation. This raises questions about the mathematical legitimacy of the eigenfunctions at the boundary.

2. Wavefunction Behavior at the Boundaries:

Because the wavefunction must be exactly zero outside the well, and finite (non-zero) inside, the functions are discontinuous in their derivatives at the boundary (if we attempt to extend the solution). This discontinuity conflicts with the requirements of the differential equation, which demands the wavefunction to be twice differentiable (except possibly at singular points).

3. Limit of Finite Depth Well:

Realistic models involve large but finite potential barriers. In these cases:

- The wavefunction penetrates slightly into the barrier.
- The eigenfunctions are continuous and differentiable everywhere.
- As the potential barrier height approaches infinity, the wavefunction's penetration depth approaches zero at the boundary.

However, in the strict infinite potential case, the wavefunction's derivative is not well-defined at the boundary, indicating a mathematical inconsistency.

Formal Mathematical Argument Against Acceptable Solutions

The core mathematical argument can be summarized as follows:

- The Schrödinger equation requires the wavefunction $\psi(x)$ to be twice differentiable within the domain.

- At the boundary where the potential jumps to infinity, the boundary condition $\psi=0$ must hold.

- However, for the solutions to satisfy the differential equation in the classical sense, the second derivative must be finite.

- In the infinite potential case, the second derivative becomes unbounded at the boundary, meaning the solution is not twice differentiable there.

This discrepancy indicates that while the solutions are mathematically constructed as eigenfunctions with certain boundary conditions, they are not physically realizable as true solutions of the Schrödinger equation with an infinitely high potential barrier. Instead, they are idealized, limiting solutions that approximate the behavior of physical systems with very high, but finite, barriers.

Physical and Philosophical Implications

The analysis above leads to profound implications:

Physical Realism of the Infinite Well

- No real physical system can have an infinite potential barrier. All barriers are finite, leading to non-zero wavefunction penetration (tunneling) into the barrier.

- The 'solutions' for the infinite well serve as idealized models—useful for gaining intuition and simplifying calculations but not exact representations of real systems.

Mathematical Idealization and Its Limits

- The infinite potential well illustrates how mathematical models can sometimes stretch the limits of physical plausibility.

- The boundary conditions imposed are mathematical conveniences that produce solutions with desirable properties (discrete energy levels, normalized wavefunctions) but can conflict with the fundamental requirements of the differential equations they arise from.

Conclusions: The Absence of Exactly Acceptable Solutions

In summary, the core reason why there is no truly acceptable (i.e., mathematically rigorous and physically realizable) solution to the time-independent Schrödinger equation for the idealized infinite potential well hinges on the incompatibility of the boundary conditions with the differential equation's requirements. The wavefunctions:

- Are constructed based on boundary conditions that force them to vanish at the boundary.

- Are not twice differentiable at the boundary because the potential jumps to infinity.

- Therefore, they are mathematically discontinuous in derivatives at the boundary, violating the conditions necessary for a true eigenfunction of the Schrödinger operator.

This insight emphasizes that the infinite potential well is an approximate idealization, useful for teaching and conceptual understanding, but fundamentally incompatible with the strict mathematical requirements of quantum mechanics. Real systems with very high but finite barriers produce solutions that are close approximations, but the exact, infinitely confined solutions are mathematically and physically unattainable.

Final Thoughts

The exploration of the infinite potential well exemplifies a broader theme in physics: idealized models are invaluable for understanding complex phenomena but have inherent limitations. Recognizing these limitations sharpens our intuition and guides us toward more realistic models that better capture the nuances of the quantum world. The acknowledgment that no truly acceptable solution exists in the strict infinite case underscores the importance of bridging idealization with physical reality—a recurring challenge and opportunity in theoretical physics.

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