State Whether The Function Is Continuous At The Indicated Point. If It Is Not Continuous, Tell Why. State

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Understanding continuity in functions is a fundamental concept in calculus and mathematical analysis. It plays a crucial role in understanding the behavior of functions, their limits, derivatives, and integrals. When analyzing whether a function is continuous at a specific point, mathematicians examine certain criteria, such as the existence of the limit at that point and the value of the function at that point. If these conditions are met, the function is continuous there; if not, it is discontinuous, and understanding the reason behind the discontinuity helps in grasping the underlying behavior of the function. This article provides an in-depth exploration of how to determine whether a function is continuous at a given point, the types of discontinuities that can occur, and practical examples illustrating the process.

Understanding Continuity of Functions

Definition of Continuity at a Point

A function $f(x)$ is said to be continuous at a point $x = a$ if the following three conditions are satisfied:

1. The function is defined at a: $f(a)$ exists.
2. The limit exists at a: $\lim_{x \to a} f(x)$ exists.
3. The limit equals the function value: $\lim_{x \to a} f(x) = f(a)$.

Mathematically, this can be summarized as:

$$
\text{f is continuous at } a \quad \Longleftrightarrow \quad \text{(1) } f(a) \text{ exists,} \quad \text{(2) } \lim_{x \to a} f(x) \text{ exists,} \quad \text{(3) } \lim_{x \to a} f(x) = f(a).
$$

How to Test for Continuity at a Point

Determining whether a given function is continuous at a specific point involves evaluating the function and its limit at that point. The following steps outline the process:

Step 1: Verify the Function is Defined at the Point

- Check whether $f(a)$ exists.
- If $f(a)$ is undefined, the function cannot be continuous at a.

Step 2: Calculate the Limit of the Function as x Approaches a

- Compute $\lim_{x \to a} f(x)$.
- This may involve:
- Direct substitution.
- Simplifying the function to evaluate the limit.
- Using limit laws or special limit techniques (e.g., L'Hôpital's rule) if direct substitution results in indeterminate forms.

Step 3: Compare the Limit and the Function Value

- If $\lim_{x \to a} f(x) = f(a)$, then the function is continuous at a.
- If not, the function is discontinuous at a.

Types of Discontinuities and Their Causes

Discontinuities are points where a function fails to be continuous. Recognizing the type of discontinuity helps in understanding the source of the problem. The main types include:

1. Removable Discontinuity

- Occurs when the limit exists at a, but $f(a)$ is either undefined or not equal to that limit.
- Often caused by a 'hole' in the graph.
- Example: $f(x) = \frac{x^2 - 1}{x - 1}$ at $x=1$. The limit exists, but the function is undefined at $x=1$.

2. Jump Discontinuity

- The limits from the left and right at a exist but are not equal.
- The function 'jumps' at the point.
- Example: Piecewise function with different definitions on either side of a.

3. Infinite Discontinuity

- The function approaches infinity (or negative infinity) as x approaches a.
- Typically occurs with vertical asymptotes.
- Example: $f(x) = \frac{1}{x}$ at $x=0$.

4. Oscillatory Discontinuity

- The function oscillates infinitely as x approaches a, preventing the limit from existing.
- Example: $\sin \frac{1}{x}$ at $x=0$.

Practical Examples and Analysis

Example 1: Polynomial Function

Suppose $f(x) = 3x^2 + 2x - 5$. Is f continuous at $x=2$?

Solution:

- Since polynomial functions are continuous everywhere, f is continuous at $x=2$.
- For confirmation:
- $f(2) = 3(2)^2 + 2(2) - 5 = 3(4) + 4 - 5 = 12 + 4 - 5 = 11$.
- $\lim_{x \to 2} f(x) = f(2) = 11$.
- Therefore, the function is continuous at $x=2$.

Example 2: Rational Function with a Hole

Let $f(x) = \frac{x^2 - 1}{x - 1}$. Is f continuous at $x=1$?

Solution:

- Check if $f(1)$ is defined:
- $f(1) = \frac{1^2 - 1}{1 - 1} = \frac{0}{0}$, indeterminate.
- Simplify the function:
- $f(x) = \frac{(x-1)(x+1)}{x-1}$.
- For $x \neq 1$, $f(x) = x + 1$.
- Compute the limit:
- $\lim_{x \to 1} f(x) = \lim_{x \to 1} x + 1 = 2$.
- Since $f(1)$ is undefined but the limit exists and equals 2, the discontinuity at $x=1$ is removable.

- To make f continuous at 1, define $f(1) = 2$.

Example 3: Piecewise Function with Jump Discontinuity

Define $f(x)$ as:

$$
f(x) = \begin{cases}
x + 2, & x < 3 \\
x - 1, & x \geq 3
\end{cases}
$$

Is f continuous at $x=3$?

Solution:

- Compute $f(3)$:
- Since $x \geq 3$, $f(3) = 3 - 1 = 2$.
- Calculate the limit from the left:
- $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x + 2 = 3 + 2 = 5$.
- Calculate the limit from the right:
- $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} x - 1 = 3 - 1 = 2$.
- Since the left limit (5) does not equal the right limit (2), the limit does not exist at $x=3$.
- Conclusion: The function is discontinuous at $x=3$ due to a jump discontinuity.

Implications of Continuity in Calculus

Continuity is a vital property for many calculus concepts:

- The Intermediate Value Theorem requires the function to be continuous on an interval.
- The existence of derivatives at a point typically requires the function to be continuous there.
- Integrability of functions over an interval often depends on the nature of discontinuities.

Understanding whether a function is continuous at a point helps in applying these theorems effectively and accurately.

Summary and Best Practices

- Always verify the function's value at the point.
- Carefully compute the limit approaching the point from both sides.
- Recognize different types of discontinuities to understand their causes.
- Use algebraic simplification, limit laws, and special techniques for complex functions.
- Be aware that functions can be continuous everywhere (like polynomials) or only at specific points.

Conclusion

Determining whether a function is continuous at a given point involves evaluating the function's value and its limit at that point. When the limit exists, and it equals the function's value, the function is continuous; otherwise, it is discontinuous. Recognizing the type of discontinuity—removable, jump, infinite, or oscillatory—provides insight into the function's behavior and potential modifications needed to make it continuous. Mastering these concepts is essential for deeper studies in calculus, differential equations, and mathematical analysis, where the behavior of functions around specific points influences the entire analysis.

Frequently Asked Questions

Is the function $f(x) = (x^2 - 4)/(x - 2)$ continuous at $x = 2$? If not, why?

The function is not continuous at $x = 2$ because it is undefined there; the denominator becomes zero, leading to a discontinuity.

Determine if $f(x) = \sqrt{x}$ is continuous at $x = 0$.

Yes, $f(x) = \sqrt{x}$ is continuous at $x = 0$ because the function is defined at 0, and the limit as x approaches 0 exists and equals f(0).

Is the step function $f(x) = 1$ if $x \geq 0$, and 0 if $x < 0$, continuous at $x = 0$?

No, it is not continuous at $x = 0$ because the left-hand limit is 0 and the right-hand limit is 1, which do not equal the function value at 0.

Determine whether $f(x) = 1/x$ is continuous at $x = 0$.

No, $f(x) = 1/x$ is not continuous at $x = 0$ because it is undefined there, resulting in a discontinuity.

Is the function f(x) = x^3 continuous at x = 1?

Yes, f(x) = x^3 is continuous at x = 1 because polynomials are continuous everywhere and the limit as x approaches 1 equals f(1).

Check if the piecewise function f(x) = { x + 2 if x < 3; 5 if x ≥ 3 } is continuous at x = 3.

It is continuous at x = 3 because the limit from the left is 3 + 2 = 5, and f(3) = 5, so the function is continuous there.

Is the function f(x) = |x| continuous at x = 0?

Yes, f(x) = |x| is continuous at x = 0 because the limit as x approaches 0 from both sides equals 0, which is f(0).

Determine if f(x) = sin(1/x) is continuous at x = 0.

No, f(x) = sin(1/x) is not continuous at x = 0 because it is undefined there; the function is not defined at x = 0.

Is the function f(x) = e^x continuous at x = -5?

Yes, f(x) = e^x is continuous at x = -5 because exponential functions are continuous everywhere, and the limit as x approaches -5 equals f(-5).

Additional Resources

Continuity: An In-Depth Analysis of Function Behavior at Specific Points

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Understanding whether a function is continuous at a particular point is fundamental in calculus and mathematical analysis. Continuity not only ensures smooth behavior of functions but also underpins many advanced concepts, such as differentiability and integrability. In this comprehensive review, we will explore the criteria for continuity, analyze common types of discontinuities, and provide detailed methods to evaluate whether a function maintains continuity at a specified point.

What Does Continuity Mean? An Expert Overview

Continuity at a point essentially means that the function behaves "without breaks" at that point. In more formal terms, a function $f(x)$ is continuous at a point $x = c$ if three conditions are met:

1. The function is defined at c: $f(c)$ exists.

2. The limit exists at c: $\lim_{x \to c} f(x)$ exists.
3. The limit equals the function value: $\lim_{x \to c} f(x) = f(c)$.

When all three conditions hold true, the function is said to be continuous at c.

Why Is Continuity Important?

- Predictability: Continuous functions have no abrupt jumps or gaps, making their behavior predictable.
- Mathematical Analysis: Many theorems (Intermediate Value Theorem, Extreme Value Theorem) require continuity.
- Practical Applications: Physical models, engineering systems, and economic theories often assume or require functions to be continuous for realistic modeling.

Criteria for Continuity at a Point

Before determining whether a function is continuous at a specific point, it’s important to understand the formal criteria:

1. Function is Defined at the Point

Ensure $f(c)$ exists. If $f(c)$ is undefined, the function cannot be continuous there.

2. Limit Exists at the Point

Calculate $\lim_{x \to c} f(x)$. The limit exists only if:

- $\lim_{x \to c^-} f(x)$ (left-hand limit) exists.
- $\lim_{x \to c^+} f(x)$ (right-hand limit) exists.
- Both limits are equal: $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$.

3. Limit Equals the Function Value

Check if $f(c) = L$. If yes, the function is continuous at c.

Common Types of Discontinuities and Their Causes

Even functions that look simple can have hidden discontinuities. Recognizing these helps diagnose why a function might fail to be continuous at a point.

1. Removable Discontinuity

- Description: Occurs when the limit exists, but $f(c)$ is either undefined or not equal to the limit.
- Cause: Usually due to a "hole" in the graph.
- Example: $f(x) = \frac{\sin x}{x}$ at $x=0$. The limit exists, but $f(0)$ is undefined unless explicitly defined.

2. Jump Discontinuity

- Description: The left and right limits exist but are not equal.
- Cause: Sudden change or "jump" in the function's value.
- Example: The step function $f(x) = \begin{cases} 1, & x<0 \\ 2, & x \geq 0 \end{cases}$ at $x=0$.

3. Infinite Discontinuity

- Description: One or both limits tend to infinity.
- Cause: Vertical asymptotes.
- Example: $f(x) = \frac{1}{x}$ at $x=0$.

4. Oscillatory Discontinuity

- Description: The limit does not exist because the function oscillates infinitely near the point.
- Example: $f(x) = \sin \frac{1}{x}$ at $x=0$.

Step-by-Step Approach to Determine Continuity at a Point

Let’s now walk through a detailed process, applying it to various types of functions and points.

Step 1: Confirm the Function is Defined at the Point

Check whether $f(c)$ exists. If not, the function is automatically discontinuous there unless the limit exists and is finite, and $f(c)$ is then defined to be that limit (removable discontinuity).

Step 2: Compute the Limits from Both Sides

Calculate:

$$
\lim_{x \to c^-} f(x) \quad \text{and} \quad \lim_{x \to c^+} f(x)
$$

- If both exist and are equal, the limit exists at c.
- If either does not exist or they are not equal, the function is discontinuous at c.

Step 3: Check if the Limit Equals the Function Value

Compare $\lim_{x \to c} f(x)$ with $f(c)$:

- If they are equal, the function is continuous at c.
- If not, the discontinuity is removable; the function can be made continuous by defining $f(c)$ equal to the limit.

Step 4: Identify the Type of Discontinuity (if any)

Based on the previous steps, classify the discontinuity:

- Removable: Limit exists, but $f(c)$ is different or undefined.
- Jump: Limits exist but are unequal.
- Infinite: Limit tends to infinity.
- Oscillatory: Limit does not exist due to oscillations.

Case Studies: Analyzing Specific Functions at Given Points

Let's examine several examples to illustrate the application of the above steps.

Example 1: Polynomial Function at a Point

Function: $f(x) = 3x^2 + 2x - 5$

Point: $x=2$

Analysis:

- The polynomial is defined everywhere, so $f(2) = 3(2)^2 + 2(2) - 5 = 3(4) + 4 - 5 = 12 + 4 - 5 = 11$.
- The limits as $x \to 2$ are straightforward: $\lim_{x \to 2} f(x) = f(2) = 11$.
- Since the function is defined at 2, and the limit equals the function value, f(x) is continuous at $x=2$.

Example 2: Rational Function with a Potential Discontinuity

Function: $f(x) = \frac{x^2 - 4}{x - 2}$

Point: $x=2$

Analysis:

- $f(2)$ is undefined because denominator zero: division by zero.
- Simplify numerator: $x^2 - 4 = (x - 2)(x + 2)$.
- Cancel common factor: $f(x) = x + 2$, for $x \neq 2$.
- Limit as $x \to 2$: $\lim_{x \to 2} f(x) = 2 + 2 = 4$.
- Since $f(2)$ is undefined, the discontinuity at $x=2$ is removable.
- To make f continuous at 2, define $f(2) = 4$.

Example 3: Step Function at a Discontinuity

Function:

$$
f(x) = \begin{cases}
1, & x < 0 \\
2, & x \geq 0
\end{cases}
$$

Point: $x=0$

Analysis:

- $f(0) = 2$.
- Left-hand limit: $\lim_{x \to 0^-} f(x) = 1$.
- Right-hand limit: $\lim_{x \to 0^+} f(x) = 2$.
- Since the limits are not equal ($1 \neq 2$), discontinuity is jump discontinuity at $x=0$.

Example 4: Function with Asymptote

Function: $f(x) = \frac{1}{x-3}$

Point: $x=3$

Analysis:

- $f(3)$ is undefined.
- $\lim_{x \to 3^-} f(x) = -\infty$, $\lim_{x \to 3^+} f(x) = +\infty$.
- The limits tend to infinity; thus, infinite discontinuity at $x=3$.

Special Cases and Considerations

While the above examples cover most situations, some functions require special attention:

- Piecewise functions: Always analyze each piece near the point of interest.
- Absolute value functions: Check whether the absolute value causes a corner

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