. Set Up And Evaluate An Integral That Computes The Arc Length Of The Curve Y = Ln (csc X) On The Interval

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Introduction

Understanding the concept of arc length is fundamental in calculus, especially when analyzing the geometry of curves. The arc length gives us the measure of the distance along a curve between two points, providing insight into the curve's shape and extent. In this article, we will explore how to set up and evaluate the integral that computes the arc length of the specific curve $y = \ln (\csc x)$ over a given interval. This process involves understanding the properties of the function, calculating its derivative, and applying the arc length formula.

Background and Fundamental Concepts

Arc Length Formula

The arc length L of a smooth curve $y = f(x)$ from $x = a$ to $x = b$ is given by the integral:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

This formula derives from the Pythagorean theorem, considering infinitesimal segments along the curve.

Understanding the Function $y = \ln (\csc x)$

The function $y = \ln (\csc x)$ involves the cosecant function, which is the reciprocal of sine:

$$
\csc x = \frac{1}{\sin x}
$$

Hence, the function can be rewritten as:

$$
y = \ln \left(\frac{1}{\sin x} \right) = - \ln (\sin x)
$$

\]

This transformation simplifies differentiation and analysis.

Determining the Interval for the Arc Length

Before setting up the integral, we need to specify the interval $[a, b]$ over which the arc length is to be computed. For the function $y = \ln (\csc x)$, the domain must be carefully considered because:

- $\sin x \neq 0$, so $x \neq n\pi$ where n is an integer.
- $\csc x$ is positive and defined for $x \in (0, \pi)$, $(\pi, 2\pi)$, etc., but the natural logarithm requires positive arguments, which aligns with $\csc x > 0$.

Suppose we are asked to compute the arc length on the interval $(0, \pi/2)$. This interval is suitable because:

- $\sin x > 0$ in $(0, \pi/2)$,
- $\csc x > 0$,
- $y = - \ln (\sin x)$ is well-defined and smooth in this range.

For demonstration, we set:

$$
a = \epsilon \quad (\text{approaching } 0^+), \quad b = \pi/2
$$

The choice of $\epsilon \to 0^+$ accounts for the behavior near zero, but for practical purposes, we often consider a small positive value.

Note: The interval can be adjusted based on the specific problem statement.

Calculating the Derivative dy/dx

To set up the arc length integral, we need dy/dx.

Given:

$$
y = - \ln (\sin x)
$$

Differentiate:

$$
\frac{dy}{dx} = - \frac{1}{\sin x} \cdot \cos x = - \cot x
$$

Thus,

$$
\left(\frac{dy}{dx} \right)^2 = \cot^2 x
$$

Important: The derivative is valid within the interval where $\sin x > 0$.

Setting Up the Arc Length Integral

Using the general formula:

$$
L = \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

Substitute $dy/dx = - \cot x$:

$$
L = \int_{a}^{b} \sqrt{1 + \cot^2 x} \, dx
$$

Recall the Pythagorean identity:

$$
1 + \cot^2 x = \csc^2 x
$$

Therefore:

$$
L = \int_{a}^{b} \sqrt{\csc^2 x} \, dx
$$

Since $\csc x > 0$ in the interval, the square root simplifies to:

$$
L = \int_{a}^{b} \csc x \, dx
$$

Thus, the arc length reduces to evaluating the integral of $\csc x$ over the interval.

Evaluating the Integral of $\csc x$

Standard Integral of $\csc x$

The integral:

$$
\int \csc x \, dx
$$

is a well-known integral with the standard result:

$$
\int \csc x \, dx = - \ln | \csc x + \cot x | + C
$$

This formula is derived through multiplying numerator and denominator by $\csc x + \cot x$ and using substitution.

Applying the Limits

Suppose we evaluate the arc length from $x = \epsilon$ to $x = \pi/2$:

$$
L = \left[- \ln | \csc x + \cot x | \right]_{\epsilon}^{\pi/2}
$$

At $x = \pi/2$:

$$
\sin (\pi/2) = 1, \quad \cos (\pi/2) = 0
$$

$$
\csc (\pi/2) = 1, \quad \cot (\pi/2) = 0
$$

So,

$$
- \ln | 1 + 0 | = - \ln 1 = 0
$$

At $x = \epsilon$:

$$
\sin \epsilon \approx \epsilon, \quad \cos \epsilon \approx 1
$$

$$
\csc \epsilon \approx \frac{1}{\epsilon}, \quad \cot \epsilon \approx \frac{1}{\epsilon}
$$

Thus,

$$
\csc \epsilon + \cot \epsilon \approx \frac{1}{\epsilon} + \frac{1}{\epsilon} = \frac{2}{\epsilon}
$$

and

$$
- \ln \left| \frac{2}{\epsilon} \right| = - \left(\ln 2 - \ln \epsilon \right) = - \ln 2 + \ln \epsilon
$$

Putting it all together:

$$
L = \left[0 \right] - \left[- \ln 2 + \ln \epsilon \right] = \ln 2 - \ln \epsilon
$$

As $\epsilon \to 0^+$, $\ln \epsilon \to -\infty$, implying the length diverges. Therefore, the arc length from 0 to $\pi/2$ is infinite, indicating the curve has an infinite arc length near $x = 0$.

Note: For finite bounds away from the singularity at $x=0$, the integral yields finite results.

Generalized Expression for the Arc Length

For a finite interval $[a, b] \subset (0, \pi/2)$, the arc length of $y = \ln (\csc x)$ is:

$$
L = \left[- \ln | \csc x + \cot x | \right]_a^b
$$

which simplifies to:

$$
L = - \ln | \csc b + \cot b | + \ln | \csc a + \cot a |
$$

or equivalently,

$$
L = \ln \left| \frac{\csc a + \cot a}{\csc b + \cot b} \right|
$$

This expression provides a straightforward way to compute the arc length over any subinterval where the function is well-defined and smooth.

Conclusion and Summary

Calculating the arc length of the curve $y = \ln (\csc x)$ involves several key steps:

1. Identify the interval over which the arc length is to be computed, ensuring the function is defined and smooth.

2. Rewrite the function to simplify differentiation: $y = - \ln (\sin x)$.

3. Differentiate to find dy/dx, obtaining $- \cot x$.

4. Set up the integral for arc length using the formula:

$$
L = \int_{a}^{b} \sqrt{1 + (\frac{dy}{dx})^2} \, dx
$$

which simplifies to:

$$
L = \int_{a}^{b} \csc x \, dx
$$

5. Evaluate the integral using the standard result

Frequently Asked Questions

How do you set up the integral to find the arc length of the curve y = ln(csc x) over a specific interval?

To set up the arc length integral, use the formula $L = \int$ from a to b of $\sqrt{(1 + (dy/dx)^2)}\ dx$, where dy/dx is the derivative of $y = \ln(\csc x)$. First, find dy/dx, then substitute into the formula and integrate over the interval [a, b].

What is the derivative of y = ln(csc x) with respect to x?

The derivative $dy/dx = -\cot x$, because $d/dx\ [\ln(\csc x)] = -\cot x$, using the chain rule and derivative of cotangent.

How do you simplify the expression inside the square root when computing the arc length for y = ln(csc x)?

Calculate $1 + (dy/dx)^2 = 1 + (-\cot x)^2 = 1 + \cot^2 x$. Since $1 + \cot^2 x = \csc^2 x$, the integrand simplifies to $\sqrt{(\csc^2 x)} = |\csc x|$.

What is the resulting integral expression for the arc length of y = ln(csc x)?

The arc length L = ∫ from a to b of |csc x| dx, since the integrand simplifies to |csc x| after substitution.

How do you evaluate the integral of |csc x| over an interval?

The integral of |csc x| dx can be evaluated by considering the interval's location relative to the x-axis, and using known integrals: ∫ csc x dx = -ln |csc x + cot x| + C. Adjust for the absolute value depending on the interval.

Are there specific intervals where the arc length integral for y = ln(csc x) simplifies further?

Yes, over intervals where csc x is positive (e.g., (0, π)), |csc x| = csc x, simplifying the integral to ∫ csc x dx. For intervals where csc x is negative, adjust the integral accordingly.

What are common challenges when evaluating the arc length integral of y = ln(csc x)?

Challenges include handling the absolute value of csc x, carefully evaluating the integral of csc x, and managing potential discontinuities or asymptotes at the interval endpoints.

How do you interpret the geometric meaning of the arc length of y = ln(csc x)?

The arc length measures the length of the curve y = ln(csc x) between two points x = a and x = b, representing the actual 'distance' along the curve on the xy-plane.

Can the integral for the arc length of y = ln(csc x) be expressed in terms of elementary functions after integration?

Yes, the integral reduces to a combination of logarithmic functions, specifically involving the integral of csc x, which is -ln |csc x + cot x|, making the arc length expressible in terms of elementary functions.

What is the importance of choosing the correct interval when computing the arc length of y = ln(csc x)?

Choosing the correct interval ensures the integrand remains well-defined and continuous. Since csc x has asymptotes at multiples of π, selecting intervals that avoid these points simplifies the calculation and interpretation.

Additional Resources

Set Up And Evaluate An Integral That Computes The Arc Length Of The Curve y = ln(csc x) On The Interval

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Introduction

Calculus provides powerful tools for understanding the geometry of curves, one of which is the arc length formula. Calculating the length of a curve over a specific interval involves setting up and evaluating an integral derived from the fundamental theorem of calculus. This process becomes particularly interesting and educational when applied to a non-trivial function such as $y = \ln(\csc x)$.

In this detailed analysis, we will explore how to set up and evaluate the integral for the arc length of the curve $y = \ln(\csc x)$ over a given interval. We will delve into the function’s properties, the necessary derivatives, the integral setup, and the evaluation process, including potential challenges and insights.

Understanding the Function $y = \ln(\csc x)$

1. Basic properties

- Domain considerations:
Since $\csc x = \frac{1}{\sin x}$, the function $y = \ln(\csc x)$ is defined where $\csc x > 0$.
- $\csc x$ is positive where $\sin x > 0$, i.e., in intervals $(0, \pi)$, $(2\pi, 3\pi)$, etc.
- For simplicity, assume our interval of interest is within $(0, \pi)$.

- Behavior at endpoints:
As $x \to 0^+$, $\sin x \to 0^+$, so $\csc x \to +\infty$, and $y \to +\infty$.
As $x \to \pi^-$, $\sin x \to 0^+$, so $\csc x \to +\infty$, and $y \to +\infty$.
Thus, the function tends to infinity at both endpoints, indicating the curve is unbounded near the interval edges.

2. Simplification of the function

Expressing y explicitly:

$$
y = \ln(\csc x) = \ln \left(\frac{1}{\sin x} \right) = -\ln(\sin x)
$$

This alternative form simplifies derivatives and calculations.

Derivative of $y = -\ln(\sin x)$

Calculating the derivative is crucial for the arc length integral:

$$
\frac{dy}{dx} = - \frac{1}{\sin x} \cdot \cos x = - \cot x
$$

Key points:

- The derivative $y' = - \cot x$ reflects the slope of the curve at any point x.
- The behavior of y' near the endpoints (where $\sin x \to 0$) influences the integrability and potential singularities.

Setting Up the Arc Length Integral

1. Arc length formula

For a function $y = f(x)$ continuous and differentiable over $[a, b]$, the arc length L is:

$$
L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \, dx
$$

Applying this to our specific function, the integral becomes:

$$
L = \int_a^b \sqrt{1 + (\! - \cot x)^2} \, dx
$$

which simplifies to:

$$
L = \int_a^b \sqrt{1 + \cot^2 x} \, dx
$$

since the square eliminates the negative sign.

2. Simplification within the integrand

Recall the Pythagorean identity:

$$
1 + \cot^2 x = \csc^2 x
$$

Thus,

$$
L = \int_a^b \sqrt{\csc^2 x} \, dx = \int_a^b |\csc x| \, dx
$$

Given the interval $(0, \pi)$, where $\sin x > 0$, $\csc x > 0$, so:

$$
L = \int_a^b \csc x \, dx
$$

This is a significant simplification: the arc length integral reduces to the integral of $\csc x$ over the interval.

Evaluating the Integral $\int \csc x \, dx$

The indefinite integral of $\csc x$ is well-known:

$$
\int \csc x \, dx = - \ln \left| \csc x + \cot x \right| + C
$$

This formula is derived using a standard technique involving multiplying numerator and denominator by $\csc x + \cot x$ to facilitate substitution.

Complete Setup for the Arc Length

Suppose the interval is $[a, b] \subset (0, \pi)$. Then, the arc length is:

$$
L = \left[- \ln \left| \csc x + \cot x \right| \right]_a^b
$$

or explicitly,

$$
L = - \ln \left| \csc b + \cot b \right| + \ln \left| \csc a + \cot a \right|
$$

which simplifies to:

$$
L = \ln \left(\frac{ \left| \csc a + \cot a \right| }{ \left| \csc b + \cot b \right| } \right)
$$

Given the positivity of $\csc x$ and $\cot x$ in the interval, the absolute value can often be dropped, assuming the interval is within the principal domain where these quantities are positive.

Practical Evaluation and Examples

To solidify understanding, let's evaluate the arc length over specific subintervals within $(0, \pi)$.

Example: From $a = \frac{\pi}{4}$ to $b = \frac{\pi}{2}$

- At $x = \pi/4$:

$$
\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}, \quad \csc \frac{\pi}{4} = \sqrt{2}
$$
$$
\cot \frac{\pi}{4} = 1
$$

- At $x = \pi/2$:

$$
\sin \frac{\pi}{2} = 1, \quad \csc \frac{\pi}{2} = 1
$$
$$
\cot \frac{\pi}{2} = 0
$$

The arc length:

$$
L = \ln \left(\frac{\left| \sqrt{2} + 1 \right|}{\left| 1 + 0 \right|} \right) = \ln (\sqrt{2} + 1)
$$

Numerically,

$$
L \approx \ln (1.4142 + 1) = \ln (2.4142) \approx 0.8814
$$

This example illustrates how the integral yields a finite value for the arc length over a finite interval within the domain.

Addressing Infinite or Unbounded Intervals

Given the behavior of $y = \ln (\csc x)$ near 0 and π, the integral from 0 to π involves limits approaching infinity:

$$
L = \lim_{a \to 0^+} \left[- \ln \left(\csc a + \cot a \right) \right] + \lim_{b \to \pi^-} \left[- \ln \left(\csc b + \cot b \right) \right]
$$

Evaluating these limits:

- As \(a \to 0^+ \):

\[
\sin a \to 0^+, \quad \csc a \to +\infty, \quad \cot a \to +\infty
\]

Thus:

\[
\csc a + \cot a \to +\infty \quad \Rightarrow \quad - \ln (\text{large}) \to -\infty
\]

- As \(b \to \pi^- \):

\[
\sin b \to 0^+, \quad \csc b \to +\infty, \quad \cot b \to -\infty
\]

So:

\[
\csc b + \cot b \to +\infty - \infty
\]

which is indeterminate. But considering the asymptotic behavior:

\[
\csc x \sim \frac{1}{\pi - x}, \quad \cot x \sim - \frac{1}{\pi - x}
\]

then:

\[
\csc x + \cot x \sim \frac{1}{\pi - x} - \frac{1}{\pi - x} = 0
\]

meaning the expression tends to zero, and:

\[
- \ln (\text{small

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