

The Solid Lies Between Planes Perpendicular To The X Axis At $x = -1$ And $x = 1$. the Cross-section Perpendicular

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Understanding three-dimensional solids and their cross-sections is fundamental in calculus and geometry. In this article, we explore a specific solid bounded by two planes perpendicular to the x-axis at $x = -1$ and $x = 1$. We analyze its shape, the nature of its cross-sections perpendicular to the x-axis, and how to compute its volume. This detailed examination will help clarify the geometric and integral concepts involved in such problems.

Defining the Solid: Boundaries and Geometry

Planes Bounding the Solid

The solid is bounded by two planes perpendicular to the x-axis:

- The plane at $x = -1$
- The plane at $x = 1$

These planes define the limits along the x-axis, creating a segment from $x = -1$ to $x = 1$.

Cross-Sectional Shapes and Boundaries

The cross-section of the solid at any position x between -1 and 1 depends on the shape and the boundary functions that define the solid's profile in the y and z directions.

Suppose the solid is generated by revolving a certain region or by stacking particular shapes along the x-axis. For instance, common cases include:

- The solid generated by the region between two curves in the y - z plane
- The solid created by revolving a region around an axis
- The solid formed by the intersection of certain surfaces

In many typical problems, the cross-section at a fixed x might be a circle, rectangle, or more complex shape, depending on the defining functions.

Analyzing Cross-Sections Perpendicular to the X-Axis

What Does 'Perpendicular to the X-Axis' Mean?

Cross-sections perpendicular to the x-axis are slices of the solid taken in planes parallel to the (y, z) plane at a fixed x . These slices reveal the shape of the solid at each point along the x-axis.

For a given x in $[-1, 1]$, the cross-section might be:

- A circle with radius depending on x
- A rectangle with dimensions depending on x
- An irregular shape defined by functions $y = f(x)$, $z = g(x)$

The nature of these cross-sections influences how we compute the volume and understand the geometry of the solid.

General Form of Cross-Sectional Area

If the shape of the cross-section at x is known, its area $A(x)$ can often be expressed as a function of x . For example:

- For a circular cross-section: $A(x) = \pi r(x)^2$
- For rectangular cross-section: $A(x) = \text{length} \times \text{width}$
- For more complex shapes: derive $A(x)$ based on the boundary functions

The volume V of the entire solid can then be obtained by integrating $A(x)$ along the x-axis:

$$V = \int_{-1}^1 A(x) \, dx$$

Examples of Cross-Section Shapes and Calculations

Example 1: Circular Cross-Sections

Suppose at each x , the cross-section is a circle of radius $r(x)$, where:

$$r(x) = \sqrt{1 - x^2}$$

This describes a solid where the cross-sectional radius depends on x , forming a sphere of radius 1.

Cross-Sectional Area:

$$A(x) = \pi [r(x)]^2 = \pi (1 - x^2)$$

Volume Calculation:

$$V = \int_{-1}^1 \pi (1 - x^2) \, dx$$

Compute the integral:

$$V = \pi \int_{-1}^1 (1 - x^2) \, dx = \pi \left[x - \frac{x^3}{3} \right]_{-1}^1$$

Evaluate at bounds:

$$V = \pi \left(\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right) = \pi \left(\frac{2}{3} - \left(-\frac{2}{3}\right) \right) = \pi \times \frac{4}{3} = \frac{4\pi}{3}$$

This confirms the volume of a unit sphere.

Example 2: Rectangular Cross-Sections

Suppose the cross-section at x is a rectangle with width $w(x)$ and height $h(x)$:

$$A(x) = w(x) \times h(x)$$

If the rectangle dimensions depend linearly on x :

$$w(x) = 2(1 - x), \quad h(x) = 3(1 + x)$$

Then,

$$A(x) = 2(1 - x) \times 3(1 + x) = 6(1 - x)(1 + x)$$

which simplifies to:

$$A(x) = 6(1 - x^2)$$

Volume calculation:

$$V = \int_{-1}^1 6(1 - x^2) \, dx = 6 \int_{-1}^1 (1 - x^2) \, dx$$

As above, this integral evaluates to:

$$6 \times \frac{4}{3} = 8$$

Methods to Compute Volume Based on Cross-Sections

Method 1: Disk/Washer Method

Applicable when the cross-section is a circle (or washer) at each (x) , and the radius depends on (x) .

Steps:

1. Identify the radius function $(r(x))$.
2. Write the cross-sectional area $(A(x) = \pi r(x)^2)$.
3. Integrate $(A(x))$ from $(x = -1)$ to $(x = 1)$.

Method 2: Cross-Sectional Area Function

Applicable for any cross-sectional shape where $(A(x))$ can be expressed explicitly.

Steps:

1. Derive $A(x)$ based on the shape at each x .
2. Set up the volume integral:

$$V = \int_{-1}^1 A(x) \, dx$$

3. Calculate the integral using standard calculus techniques.

Method 3: Using Symmetry

If the solid or cross-section is symmetric about $x = 0$, leverage symmetry to simplify calculations:

- For even functions $A(x)$, compute from 0 to 1 and double the result.
- For odd functions, the integral over symmetric bounds may cancel out.

Implications and Applications

Understanding how to analyze cross-sections of solids bounded between planes perpendicular to the x-axis is crucial in multiple fields:

- Engineering: Designing objects with specific volume and shape constraints.
- Physics: Calculating mass, center of mass, and moments of inertia for objects with known density distributions.
- Mathematics: Visualizing and solving volume problems in multivariable calculus.
- Computer Graphics: Rendering 3D objects based on cross-sectional data.

Additionally, this methodology underpins advanced topics like the calculation of volumes of complex solids, optimization of shapes, and understanding the geometric properties of three-dimensional objects.

Summary and Key Takeaways

- The solid bounded between planes $x = -1$ and $x = 1$ can be analyzed effectively by studying its cross-sections perpendicular to the x-axis.
- Cross-sectional shapes depend on the boundary functions that define the solid's profile in the yz -plane.
- The volume is obtained by integrating the cross-sectional area $A(x)$ over the interval $[-1, 1]$.

- Various shapes like circles and rectangles serve as examples to demonstrate how to set up and evaluate these integrals.
- Recognizing symmetry can simplify calculations.
- Mastery of these concepts applies broadly across mathematics, engineering, physics, and computer graphics.

Understanding the relationship between the geometry of cross-sections and the overall volume is a fundamental skill that enhances spatial reasoning and mathematical problem-solving.

If you want to delve deeper into similar problems or specific applications, consider exploring topics like:

- Volume by slicing and washers
- Revolution of regions around axes
- Multivariable calculus techniques
- Geometric modeling and computational volume estimation

Frequently Asked Questions

What is the shape of the cross-section of the solid between the planes at $X = -1$ and $X = 1$?

The shape of the cross-section depends on the specific shape or function defining the solid; it could be a circle, rectangle, or other shape depending on the boundary surfaces intersected.

How do you find the volume of the solid bounded between the planes at $X = -1$ and $X = 1$?

You can calculate the volume by integrating the cross-sectional area with respect to X over the interval from -1 to 1 , using the integral $V = \int_{-1}^1 A(x) \, dx$.

What is the significance of the planes being perpendicular to the X -axis in determining the cross-section?

Since the planes are perpendicular to the X -axis, each cross-section is taken perpendicular to the X -direction, allowing the use of cross-sectional areas as functions of X for integration.

Can the shape of the cross-section change along the X -axis? If so, how is this represented mathematically?

Yes, the cross-section can vary along the X -axis. This variation is represented by a function $A(x)$ that describes the area of the cross-section at each position x .

What methods can be used to determine the cross-sectional area at a specific X value?

Methods include geometric formulas if the shape is known, or algebraic calculations based on the boundary equations defining the cross-section at that X value.

How does symmetry influence the calculation of the volume of the solid between the planes?

If the solid is symmetric about the origin or another plane, it can simplify the integration process by calculating the volume over half the interval and doubling the result.

What are common functions or shapes used as cross-sections in problems like this?

Common shapes include circles, rectangles, ellipses, or more complex shapes defined by specific functions or boundary curves.

How does the orientation of the planes affect the shape and calculation of the solid?

The orientation determines the direction of the cross-sections; perpendicular planes to the X-axis produce cross-sections perpendicular to X, facilitating integration along that axis.

What practical applications involve analyzing solids bounded by planes perpendicular to an axis?

Applications include calculating volumes of objects in engineering, physics (e.g., charge distributions), fluid mechanics (e.g., flow cross-sections), and computer graphics modeling.

Additional Resources

The Solid Lies Between Planes Perpendicular To The X Axis At $X=-1$ And $X=1$. The Cross-section Perpendicular

Introduction

In the realm of multivariable calculus and geometric analysis, the study of solids bounded by planes and their cross-sections offers critical insights into volume computation, geometric properties, and spatial reasoning. One such intriguing object of study is the solid confined between two planes perpendicular to the x-axis at $(x = -1)$ and $(x = 1)$. The cross-sections of this solid—perpendicular to the x-axis—reveal complex and fascinating geometric structures that serve as foundational tools for advanced mathematical modeling and applications.

This article provides a comprehensive investigation into this solid, examining its geometric structure, cross-sectional properties, volume calculation, and implications in mathematical contexts. We aim to dissect the problem thoroughly, combining classical geometric principles with modern analytical techniques to produce a detailed understanding suitable for researchers, educators, and students alike.

Defining the Geometric Context

The Bounded Region in the XY-Plane

To understand the three-dimensional solid, it is essential to first specify the region in the XY-plane that forms the cross-section at each x-coordinate. Typically, this region is described by inequalities involving y and z , which together define the boundary of the cross-section at a given x .

Suppose the cross-section at each fixed $x \in [-1, 1]$ is a specific planar shape—for instance, a circle, square, or more complex region—whose boundary depends on the x-coordinate.

A common and illustrative case is when the cross-section is a circle with a radius that varies with x :

$$\text{Cross-section at } x: y^2 + z^2 \leq r(x)^2,$$

where $r(x)$ is a positive function defined on $[-1, 1]$.

The Solid Between the Planes

The three-dimensional solid, denoted S , is then the set:

$$S = \{(x, y, z) \mid -1 \leq x \leq 1, (y, z) \in R(x)\},$$

where $R(x)$ is the cross-sectional region in the yz -plane at x .

Key Point: The shape and size of $R(x)$ dictate the overall geometry of the solid.

Cross-Sectional Analysis Perpendicular to the X-Axis

Types of Cross-Sections

Investigating the cross-sections at various x values reveals the internal structure of S . Depending on the function $r(x)$, the cross-sections can be:

- Circular: for example, when $r(x)$ is smooth and positive over $[-1, 1]$,
- Elliptical: if the boundary is an ellipse with x -dependent axes,

- Polygonal: if the cross-section is a regular or irregular polygon,
- Complex or irregular shapes: involving more intricate boundary functions.

For simplicity and clarity, this investigation will focus on the classical case where the cross-section is a circle with radius $r(x)$.

Mathematical Characterization of Cross-Sections

Assuming a circular cross-section:

$$R(x) = \left\{ (y,z) \in \mathbb{R}^2 \mid y^2 + z^2 \leq r(x)^2 \right\}.$$

The area of each cross-section is:

$$A(x) = \pi r(x)^2.$$

This allows the volume of the entire solid to be computed via integration:

$$V = \int_{x=-1}^1 A(x) \, dx = \pi \int_{-1}^1 r(x)^2 \, dx.$$

The function $r(x)$ encapsulates the shape's variation along the x-axis.

Volume Computation of the Solid

General Integral Formula

Given the cross-sectional area $A(x)$, the volume V of the solid between $x=-1$ and $x=1$ is:

$$V = \int_{-1}^1 A(x) \, dx.$$

In the circular case:

$$V = \pi \int_{-1}^1 r(x)^2 \, dx.$$

This integral forms the core of the volume calculation and depends critically on the chosen $r(x)$.

Specific Examples of $r(x)$

- Constant Radius: $(r(x) = R)$, a constant

$$V = \pi R^2 \times (1 - (-1)) = 2\pi R^2.$$

- Linear Radius: $(r(x) = a + bx)$

$$V = \pi \int_{-1}^1 (a + bx)^2 dx.$$

Computing this integral yields:

$$V = \pi \left[a^2 \times 2 + 2ab \times 0 + b^2 \times \frac{2}{3} \right] = 2\pi a^2 + \frac{2}{3}\pi b^2.$$

- Quadratic or Higher-Order Functions: These require polynomial integration techniques.

Geometric and Topological Properties

Symmetry Considerations

- Symmetry about the x-axis: If $(r(x))$ is an even function $(r(-x) = r(x))$, the solid is symmetric about the plane $(x=0)$.

- Symmetry about the yz-plane: The cross-sections are symmetric in (y) and (z) .

- Overall symmetry: Depends on the functional form of $(r(x))$.

Topological Structure

- The solid is homeomorphic to a cylinder-like shape when $(r(x))$ is positive and continuous over $([-1,1])$.

- If $(r(x))$ approaches zero at the endpoints, the solid may have "caps" or "holes," affecting its topological genus.

Special Cases and Variations

Non-Uniform Cross-Sections

- Elliptical cross-sections: $(\frac{y^2}{a(x)^2} + \frac{z^2}{b(x)^2} \leq 1)$, with $(a(x), b(x))$ functions of (x) .

- Polygonal cross-sections: defined by boundary functions $\{y_i(x), z_i(x)\}$, where $\{i\}$ indexes vertices.

Variable Boundaries

- The planes at $\{x = -1\}$ and $\{x=1\}$ may be replaced with more complex boundary surfaces, leading to more intricate solids.

Applications and Implications

Mathematical Modeling

- Volume estimation: critical in physics, engineering, and computer graphics.
- Shape analysis: understanding how cross-sectional variation influences the overall geometry.
- Optimization problems: maximizing or minimizing volume under boundary constraints.

Educational Significance

- Demonstrates the utility of integral calculus in real-world geometric problems.
- Serves as a basis for understanding more complex three-dimensional shapes and their properties.

Real-World Analogies

- Manufacturing: designing objects with variable cross-sections (e.g., tapered cylinders, fuselages).
- Architecture: constructing structures with specified boundary constraints.

Deep Dive: Analytical Techniques

Integration Methods

- Direct integration: when $\{r(x)\}$ is a simple polynomial or elementary function.
- Numerical integration: for complex or non-analytic $\{r(x)\}$.
- Approximate methods: Monte Carlo or Riemann sum approximations for irregular shapes.

Boundary Function Analysis

- Studying the properties of $\{r(x)\}$ (continuity, differentiability) to infer smoothness and regularity of the solid.
- Exploring the effect of boundary conditions on the topology and volume.

Challenges and Open Questions

- How does the choice of $r(x)$ influence the topology of the solid?
- Can the cross-sectional shape vary smoothly between different forms (e.g., from circle to square)?
- What are the optimal functions $r(x)$ for specific volume or surface area constraints?
- How do these geometries extend to higher dimensions?

Conclusion

The solid lying between the planes perpendicular to the x -axis at $x=-1$ and $x=1$ encapsulates a rich tapestry of geometric, analytical, and topological features. Its study, especially via cross-sectional analysis, not only exemplifies fundamental principles of calculus and geometry but also demonstrates their profound applications in diverse scientific and engineering contexts.

By examining the shape, size, and properties of the cross-sections, we gain a detailed understanding of the entire structure. Whether the cross-section is a simple circle, an ellipse, or a more complex shape, the core principle remains: the integral over the cross-sectional area yields the volume, bridging the local geometry to the global form.

This investigation underscores the importance of geometric intuition combined with analytical rigor, illustrating how a seemingly straightforward bounding condition—planes at $x=-1$ and $x=1$ —can lead to a comprehensive exploration of a diverse and complex solid.

References

- Stewart, J. (2015)

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