Use Appropriate Algebra And Theorem 7.2.1 To Find The Given Inverse Laplace Transform. (Write Your Answer

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Introduction

The process of finding the inverse Laplace transform is a fundamental technique in solving differential equations, control systems, and various engineering problems. It involves transforming a complex algebraic expression in the Laplace domain back into the time domain, providing insights into the behavior of systems over time. To accurately compute the inverse Laplace transform, mathematicians and engineers rely on a combination of algebraic manipulation and established theorems, notably Theorem 7.2.1. This article aims to guide you through these methods, demonstrating how to effectively use algebra and Theorem 7.2.1 to find the inverse Laplace transform of a given function.

Understanding the Importance of Algebra in Inverse Laplace Transforms

Before applying any theorems, it is crucial to manipulate the given Laplace-transformed function algebraically. Proper algebraic techniques simplify complex expressions, making the application of inverse transformation formulas more straightforward.

Key Algebraic Techniques

- Partial Fraction Decomposition: Breaking down complex rational functions into simpler fractions that correspond to known inverse Laplace transforms.
- Factorization: Factoring the denominator to identify distinct roots, which correspond to standard inverse transforms.
- Completing the Square: Used when dealing with quadratic expressions in the denominator to match standard forms involving exponentials and sine/cosine functions.
- Simplification: Combining like terms and reducing expressions to their simplest form to facilitate easier application of the inverse transform.

The Role of Theorem 7.2.1 in Inverse Laplace Transforms

Theorem 7.2.1 is a pivotal result in the theory of Laplace transforms, often relating to the shifting properties or specific inversion formulas for standard functions. While the exact statement of Theorem 7.2.1 can vary depending on the textbook, it generally encompasses the following ideas:

- Linearity of the Laplace Transform: The inverse transform of a sum is the sum of the inverse transforms.
- Shifting Theorems: How multiplication by exponential functions in the Laplace domain corresponds to shifts in the time domain.
- Standard Inverse Transforms: Using known inverse transforms for functions like $\frac{1}{s-a}$, $\frac{s}{s^2 + a^2}$, etc.
- Convolution Theorem: The inverse of a product in the Laplace domain corresponds to the convolution in the time domain.

Using Theorem 7.2.1 effectively involves recognizing these properties and applying them to transform complex expressions into simpler, recognizable forms.

Step-by-Step Process for Finding the Inverse Laplace Transform

1. Identify the Given Function

Suppose you are given a function $F(s)$ in the Laplace domain. Your goal is to find $f(t)$, where:

$$
f(t) = \mathcal{L}^{-1}\{F(s)\}
$$

2. Algebraic Manipulation

- Factor the denominator to find roots or to facilitate partial fraction decomposition.
- Express the numerator in terms matching the derivatives or standard forms, if needed.
- Decompose the function into simpler fractions using partial fractions, ensuring each denominator matches a standard inverse transform.

3. Apply Theorem 7.2.1 and Known Inverse Transforms

- Use the theorem to handle terms involving shifts, exponentials, or convolutions.
- Match each term in the partial fractions to recognized inverse Laplace transforms from standard tables.

4. Combine Results

- Sum the inverse transforms of each partial fraction to obtain the final answer.
- Simplify the expression as needed for clarity and ease of interpretation.

Practical Example: Applying Algebra and Theorem 7.2.1

Let's consider an example to illustrate these steps.

Given Function:

$$
F(s) = \frac{3s + 4}{(s + 1)(s^2 + 4)}
$$

Step 1: Algebraic Manipulation

- Partial Fraction Decomposition:

$$
\frac{3s + 4}{(s + 1)(s^2 + 4)} = \frac{A}{s + 1} + \frac{Bs + C}{s^2 + 4}
$$

Multiply both sides by the denominator:

$$
3s + 4 = A(s^2 + 4) + (Bs + C)(s + 1)
$$

Expand:

$$
3s + 4 = A s^2 + 4A + B s (s + 1) + C (s + 1)
$$
$$
= A s^2 + 4A + B s^2 + B s + C s + C
$$

Group like terms:

$$
(A s^2 + B s^2) + (B s + C s) + (4A + C) = (A + B) s^2 + (B + C) s + (4A + C)
$$

Set equal to the original numerator:

$$
3s + 4 = (A + B) s^2 + (B + C) s + (4A + C)
$$

Matching coefficients:

- For s^2:

$$
A + B = 0
$$

- For s:

$$
B + C = 3
$$

- Constant term:

$$
4A + C = 4
$$

Solve this system:

From the first:

$$
B = -A
$$

Substitute into the second:

$$
- A + C = 3 \Rightarrow C = 3 + A
$$

Substitute into the third:

$$
4A + (3 + A) = 4 \Rightarrow 4A + 3 + A = 4 \Rightarrow 5A + 3 = 4
$$

$$
5A = 1 \Rightarrow A = \frac{1}{5}
$$

Calculate B:

$$
B = -A = -\frac{1}{5}
$$

Calculate C:

$$
C = 3 + \frac{1}{5} = \frac{15}{5} + \frac{1}{5} = \frac{16}{5}
$$

Step 2: Write Partial Fractions

$$
F(s) = \frac{\frac{1}{5}}{s + 1} + \frac{-\frac{1}{5}s + \frac{16}{5}}{s^2 + 4}
$$

Step 3: Apply Theorem 7.2.1 and Inverse Transforms

Recall standard inverse Laplace transforms:

- $\mathcal{L}^{-1} \left\{ \frac{1}{s + a} \right\} = e^{-a t}$
- $\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + \omega^2} \right\} = \cos \omega t$
- $\mathcal{L}^{-1} \left\{ \frac{\omega}{s^2 + \omega^2} \right\} = \sin \omega t$

Express the second term to match these forms:

$$
\frac{-\frac{1}{5}s + \frac{16}{5}}{s^2 + 4} = -\frac{1}{5} \cdot \frac{s}{s^2 + 4} + \frac{16/5}{s^2 + 4}
$$

Using inverse transforms:

- $\mathcal{L}^{-1} \left\{ \frac{s}{s^2 + 4} \right\} = \cos 2t$
- $\mathcal{L}^{-1} \left\{ \frac{2}{s^2 + 4} \right\} = \sin 2t$

Note that:

$$
\frac{16/5}{s^2 + 4} = \frac{8}{s^2 + 4}
$$

since $\frac{16/5}{s^2 + 4} = \frac{8}{s^2 + 4}$.

Thus, the inverse transforms:

$$
\mathcal{L}^{-1} \left\{ -\frac{1}{5} \frac{s}{s^2 + 4} \right\} = -\frac{1}{5} \cos 2t
$$
$$
\mathcal{L}^{-1} \left\{ \frac{8}{s^2 + 4} \right\} = 4 \sin 2t
$$

because:

$$
\mathcal{L}^{-1} \left\{ \frac{\omega}{s^2 + \omega^2} \right\} = \sin \omega t
$$

and here, $\omega = 2$, so:

$$
\frac{8}{s^2 + 4} = 4 \cdot \frac{2}{s^2 + 4}
$$

which corresponds to $4 \sin 2t$.

Final Expression for $f(t)$:

Frequently Asked Questions

What is the main purpose of Theorem 7.2.1 in finding inverse Laplace transforms?

Theorem 7.2.1 provides a method to find the inverse Laplace transform of functions expressed as algebraic manipulations, often involving partial fractions, by relating them to known transforms, simplifying the process.

How do you use algebraic techniques to simplify the Laplace transform before applying Theorem 7.2.1?

You factor the denominator, decompose the expression into partial fractions, and rewrite it into a form that matches standard inverse Laplace transforms, making it easier to apply Theorem 7.2.1.

Can you explain the step-by-step process of applying Theorem 7.2.1 to find an inverse Laplace transform?

First, express the Laplace transform as a rational function. Next, factor and decompose into partial fractions. Then, identify each term with a standard inverse Laplace transform using Theorem 7.2.1. Finally, sum the inverse transforms to obtain the original function.

What common algebraic pitfalls should be avoided when using Theorem 7.2.1 for inverse Laplace transforms?

Avoid incorrect factoring, neglecting to decompose into partial fractions properly, and mismatching terms with standard transforms. Also, ensure all algebraic manipulations are valid within the domain of the functions involved.

How does Theorem 7.2.1 facilitate finding inverse Laplace transforms of rational functions with repeated roots?

Theorem 7.2.1 allows the use of known inverse transforms for repeated roots by expressing the partial fractions accordingly, often involving polynomial terms multiplied by exponential functions, which are then directly invertible.

Why is it important to write your answer carefully after applying Theorem 7.2.1 to find the inverse Laplace transform?

Writing the answer clearly ensures correctness, makes it easier to verify each step, and helps communicate the solution effectively, especially when dealing with complex algebraic manipulations and multiple inverse transform components.

Additional Resources

Use Appropriate Algebra and Theorem 7.2.1 to Find the Given Inverse Laplace Transform

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Introduction

The process of finding inverse Laplace transforms is fundamental in solving differential equations, especially linear ordinary differential equations with constant coefficients. The Laplace transform converts differential equations into algebraic equations, which are often easier to manipulate and solve. Once the algebraic solution in the complex domain (s-domain) is obtained, the inverse Laplace transform returns the solution to the time domain.

In this detailed review, we will explore how to effectively use algebraic techniques, combined with Theorem 7.2.1, to find inverse Laplace transforms. We will dissect the algebraic strategies, the application of Theorem 7.2.1, and illustrate the process with examples to ensure clarity and comprehensive understanding.

Understanding the Laplace Transform and Its Inverse

Laplace Transform Basics

The Laplace transform $\mathcal{L}\{f(t)\} = F(s)$ is defined as:

$$
F(s) = \int_{0}^{\infty} e^{-st}f(t)dt,
$$

where $f(t)$ is a function defined for $t \geq 0$. The transform turns differential equations into algebraic equations in s. The inverse Laplace transform, $\mathcal{L}^{-1}\{F(s)\}$, retrieves the original function $f(t)$.

Why Algebra and Theorem 7.2.1 are Critical

- Algebraic manipulation simplifies complex $F(s)$ into recognizable forms.
- Theorem 7.2.1 provides a systematic way to invert functions that are in the form of rational functions, often encountered in practice.

Algebraic Techniques in Inverse Laplace Transform

Effective algebraic manipulation is crucial for simplifying $F(s)$ before applying the inverse transform. Here are key techniques:

1. Partial Fraction Decomposition

This method breaks down complex rational functions into simpler fractions whose inverse transforms are known or easily identified.

Steps:

- Ensure the degree of the numerator is less than the degree of the denominator.
- Factor the denominator into linear or irreducible quadratic factors.
- Express $F(s)$ as a sum of partial fractions with unknown coefficients.
- Solve for these coefficients.

Example:

Suppose $F(s) = \frac{3s + 5}{(s + 2)(s + 3)}$.

Decompose as:

$$
\frac{3s + 5}{(s + 2)(s + 3)} = \frac{A}{s + 2} + \frac{B}{s + 3}
$$

Solve for A and B, then find inverse transforms using known formulas.

2. Polynomial Division

When the degree of the numerator exceeds or equals the denominator, polynomial division simplifies $F(s)$ into a polynomial plus a proper fraction, making partial fractions applicable.

3. Recognizing Standard Forms

Certain functions have well-known inverse transforms, such as:

- $\frac{1}{s - a} \longleftrightarrow e^{at}$,
- $\frac{1}{(s - a)^n} \longleftrightarrow \frac{t^{n-1}}{(n-1)!} e^{at}$,
- $\frac{s}{s^2 + a^2} \longleftrightarrow \cos at$,
- $\frac{a}{s^2 + a^2} \longleftrightarrow \sin at$.

Recognizing these aids quick inversion after algebraic simplification.

Theorem 7.2.1 and Its Role in Inverse Laplace Transform

Statement of Theorem 7.2.1:

Suppose $F(s)$ can be written in the form $F(s) = \frac{N(s)}{D(s)}$, where $N(s)$ and $D(s)$ are polynomials, and the roots of $D(s)$ are simple (distinct). Then, the inverse Laplace transform $f(t)$ can be expressed as:

$$
f(t) = \sum_{k} \operatorname{Res}_{s=s_k} \left[e^{st} F(s) \right],
$$

where s_k are the poles (roots of $D(s)$).

In simpler terms, this theorem leverages the residue theorem from complex analysis, allowing us to evaluate the inverse transform by calculating residues at the poles of $F(s)$.

Applying Theorem 7.2.1 Step-by-Step

Step 1: Factor the Denominator

Find the roots s_k of the denominator $D(s)$. These roots are the poles of $F(s)$.

Step 2: Find the Residues

Calculate the residues at each pole:

$$
\operatorname{Res}_{s=s_k} \left[e^{st} F(s) \right].
$$

For simple poles, this simplifies to:

$$
\lim_{s \to s_k} (s - s_k) e^{st} F(s).
$$

Step 3: Sum the Residues

Sum all residues to obtain $f(t)$.

Practical Example

Let us consider an example to illustrate the entire process:

Given:

$$
F(s) = \frac{5s + 2}{(s + 1)(s + 4)}.
$$

Objective:

Find the inverse Laplace transform $f(t)$.

Detailed Solution

Step 1: Factor the denominator

It's already factored:

$$
D(s) = (s + 1)(s + 4),
$$

with roots $s = -1$ and $s = -4$.

Step 2: Partial fraction decomposition

Express $F(s)$ as:

$$
\frac{5s + 2}{(s + 1)(s + 4)} = \frac{A}{s + 1} + \frac{B}{s + 4}.
$$

Multiply both sides by $(s + 1)(s + 4)$:

$$
5s + 2 = A(s + 4) + B(s + 1).
$$

Set $s = -1$:

$$
5(-1) + 2 = A(-1 + 4) + B(0) \Rightarrow -5 + 2 = A(3) \Rightarrow -3 = 3A \Rightarrow A = -1.
$$

Set $s = -4$:

$$
5(-4) + 2 = A(0) + B(-4 + 1) \Rightarrow -20 + 2 = B(-3) \Rightarrow -18 = -3B \Rightarrow B = 6.
$$

Step 3: Write the partial fractions

$$
F(s) = \frac{-1}{s + 1} + \frac{6}{s + 4}.
$$

Step 4: Recognize inverse transforms

Using standard Laplace transform pairs:

- $\mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\} = e^{-at}$.

Thus:

$$
f(t) = - e^{-t} + 6 e^{-4t}.
$$

Alternative Approach Using Theorem 7.2.1

While the partial fraction method is standard, Theorem 7.2.1 allows an alternative approach by directly computing residues.

Residue at $s = -1$:

$$
\operatorname{Res}_{s=-1} \left[e^{st} F(s) \right] = \lim_{s \to -1} (s + 1) e^{st} \frac{5s + 2}{(s + 1)(s + 4)} = \lim_{s \to -1} e^{st} \frac{5s + 2}{s + 4}.

$$

Evaluate at $s = -1$:

$$
e^{-t} \frac{5(-1) + 2}{-1 + 4} = e^{-t} \frac{-5 + 2}{3} = e^{-t} \frac{-3}{3} = -e^{-t}.
$$

Residue at $s = -4$:

$$
\operatorname{Res}_{s=-4} \left[e^{st} F(s) \right] = \lim_{s \to -4} (s + 4) e^{st} \frac{5s + 2}{(s + 1)(s + 4)} = \lim_{s \to -4} e^{st} \frac{5s + 2}{s + 1}.
$$

At $s = -4$:

$$
e^{-4 t} \frac{5(-4) + 2}{-4 + 1} = e^{-4 t} \frac{-20 + 2}{-3} = e^{-4 t} \frac{-18}{-3} = 6 e^{-4 t}.
$$

Summing the residues:

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