Use The Disk Method Or The Shell Method To Find The Volume Of The Solid Generated By Revolving The Region

Use The Disk Method Or The Shell Method To Find The Volume Of The Solid Generated By Revolving The Region

Understanding how to calculate the volume of a solid generated by revolving a region around a given axis is a fundamental skill in calculus, especially in integral calculus and applications. Two primary techniques used for this purpose are the Disk Method and the Shell Method. Both methods transform a two-dimensional region into a three-dimensional solid by revolving it around an axis, but they approach the problem differently, and choosing the appropriate method depends on the geometry of the region and the axis of revolution.

This article provides an in-depth exploration of these methods, including when and how to use them, step-by-step procedures, and illustrative examples to help you master the concepts and apply them effectively in solving volume problems.

Understanding the Volume of Revolution

Before delving into the specific methods, it's crucial to comprehend what it means to generate a solid of revolution.

Definition:
Revolving a region in the plane about an axis creates a three-dimensional solid. The volume of this solid can be computed using integral calculus, based on the shape of the region and the axis of rotation.

Common Axes of Revolution:
- The x-axis
- The y-axis
- Vertical lines (e.g., $x = a$)
- Horizontal lines (e.g., $y = b$)
- Any arbitrary line

The choice of axis significantly influences which method is more convenient and straightforward to apply.

The Disk Method

What Is the Disk Method?

The Disk Method involves slicing the solid perpendicular to the axis of revolution, resulting in circular cross-sections (disks). The volume is then obtained by integrating the areas of these disks along the axis.

Ideal For:
- Regions revolved around the x-axis or y-axis
- Functions where the cross-sectional slices are circular disks

Steps to Apply the Disk Method

1. Identify the region and axis of revolution:
Determine the region bounded by the curves, and the axis about which it is revolved.

2. Set up the integral:
- For revolution around the x-axis (vertical slices):
$$ V = \pi \int_{a}^{b} [R(x)]^2 dx $$
- For revolution around the y-axis (horizontal slices):
$$ V = \pi \int_{c}^{d} [R(y)]^2 dy $$

3. Determine the radius function $R(x)$ or $R(y)$:
The radius is typically the distance from the axis of revolution to the curve.

4. Evaluate the integral:
Compute the definite integral to find the volume.

Example of the Disk Method

Suppose the region bounded by $y = \sqrt{x}$, $x=0$, and $x=4$ is revolved around the x-axis.

- Step 1: The region is between $x=0$ and $x=4$, bounded above by $y=\sqrt{x}$.
- Step 2: Since we're revolving around the x-axis, slices perpendicular to the x-axis are disks.

- Step 3: The radius of each disk at x is $R(x) = \sqrt{x}$.
- Step 4: Set up the integral:
$$V = \pi \int_{0}^{4} (\sqrt{x})^2 dx = \pi \int_{0}^{4} x dx$$
- Step 5: Compute:
$$V = \pi \left[\frac{x^2}{2} \right]_0^4 = \pi \left(\frac{16}{2} \right) = 8\pi$$

Thus, the volume of the solid is 8π.

The Shell Method

What Is the Shell Method?

The Shell Method involves slicing the solid parallel to the axis of revolution, resulting in cylindrical shells. The volume is obtained by integrating the lateral surface areas of these shells along the relevant dimension.

Ideal For:
- Regions revolved around vertical lines when integrating with respect to y (or vice versa)
- When the shell method simplifies the integral compared to the disk method

Steps to Apply the Shell Method

1. Identify the region and axis of revolution:
Understand the shape and boundaries of the region.

2. Set up the integral:
- For revolution around the y-axis (vertical shells):
$$V = 2\pi \int_{a}^{b} x \cdot h(x) dx$$
- For revolution around the x-axis (horizontal shells):
$$V = 2\pi \int_{c}^{d} y \cdot h(y) dy$$

3. Determine the height $h(x)$ or $h(y)$:
The height of the shell at a given value corresponds to the length of the region in that dimension.

4. Calculate the integral:
Carry out the integration to find the volume.

Example of the Shell Method

Using the same region as above, $y=\sqrt{x}$, revolved around the y-axis:

- Step 1: The boundary is between $x=0$ and $x=4$.
- Step 2: For shells, the radius is x, and the height is $y = \sqrt{x}$.
- Step 3: Set up the integral:
$$V = 2\pi \int_{0}^{4} x \cdot \sqrt{x} dx = 2\pi \int_{0}^{4} x^{3/2} dx$$
- Step 4: Evaluate:
$$V = 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^4 = 2\pi \left(\frac{2}{5} \times 4^{5/2} \right)$$

Calculate $4^{5/2}$:
$$4^{5/2} = (4^{1/2})^5 = 2^5 = 32$$

Therefore,
$$V = 2\pi \times \frac{2}{5} \times 32 = 2\pi \times \frac{64}{5} = \frac{128\pi}{5}$$

The volume is $\frac{128\pi}{5}$.

Choosing Between the Disk and Shell Method

The decision on which method to use largely depends on the orientation of the region and the axis of revolution.

Criterion	Use the Disk Method	Use the Shell Method
Axis of revolution	Horizontal (x-axis) or vertical (y-axis)	Vertical line or horizontal line
Region bounded by functions	Functions expressed in terms of the axis of revolution	Functions expressed in terms of the variable perpendicular to the axis
Simplicity of integrals	When the radius is easily expressed as a function of x or y	When the height of shells is easier to express

Tip:
If the region is bounded between curves expressed as functions of x, and the axis is horizontal, the disk method may be preferable. Conversely, if the region is better described in terms of y or the shell's radius is straightforward, the shell method may be more efficient.

Common Applications and Practice Problems

To deepen your understanding, here are common scenarios and practice tips:

- Revolving around the x-axis: Use the disk method if the region is between functions $y=f(x)$ and $y=0$.

- Revolving around the y-axis: Use the shell method if the region is between $x=a$ and $x=b$.

- Revolving around vertical lines other than the y-axis: Shell method often simplifies the integral.

- Revolving around horizontal lines other than the x-axis: Shell method is typically advantageous.

Practice Problem:
Find the volume of the solid generated by revolving the region bounded by $y=x^2$ and $y=4$ around $y=4$.

Solution Hint:
- Since the axis is $y=4$, consider the shell method because shells can be taken parallel to the axis of revolution, simplifying the integral.

Conclusion

Mastering the disk and shell methods is essential for calculating volumes of solids of revolution in calculus. Each method has its strengths, and choosing the correct one depends on the problem's geometry and the axis of revolution. Understanding the setup, the integral expressions, and the geometric intuition behind these techniques enables you to solve complex volume problems efficiently.

Key Takeaways:

- Use the Disk Method when slices perpendicular to the axis form disks, especially when functions are expressed as $y=f(x)$ or $x=g(y)$.

- Use the Shell Method when slicing parallel to the axis creates cylindrical shells, often simplifying the integral when the region is easier to describe in terms of the variable perpendicular to the axis.

By

Frequently Asked Questions

What is the main difference between the disk method and the shell method when calculating the volume of a solid of revolution?

The disk method involves integrating cross-sectional disks perpendicular to the axis of rotation, while the shell method involves integrating cylindrical shells parallel to the axis of rotation. The choice depends on the axis of revolution and the region's orientation.

When should I prefer the shell method over the disk method?

Use the shell method when the region is easier to describe with respect to the axis of rotation, especially if the region is bounded vertically and revolved around a vertical axis, or when integrating with respect to x is simpler.

How do I set up the integral for the disk method?

Identify the axis of rotation, slice the region perpendicular to that axis, and integrate the area of each disk (π times the radius squared) along the axis of revolution: Volume = $\int \pi\, [\text{radius}(x)]^2\, dx$.

How do I set up the integral for the shell method?

Identify the axis of revolution, consider cylindrical shells parallel to that axis, and integrate the lateral surface area (2π times radius times height) along the variable: Volume = $\int 2\pi\, (\text{radius})(\text{height})\, dx$ or dy.

Can both the disk and shell methods be used for the same problem? Which one is more efficient?

Yes, both methods can sometimes be used. The more efficient method depends on the region's shape and the axis of rotation. Typically, choose the method that simplifies the integral calculation.

How do I decide whether to integrate with respect to x or y?

Choose the variable that makes the boundaries of your region easier to express and the integral simpler. For example, if the region is bounded by y-functions, integrate with respect to y; if bounded by x-functions, integrate with respect to x.

What are common mistakes to avoid when applying the shell method?

Common mistakes include mixing variables for radius and height, incorrect bounds of integration, and forgetting to multiply by 2π for the shell's circumference. Carefully identify the shell's dimensions and limits.

How do I find the volume of the solid generated by revolving a region around the y-axis using the shell method?

Set up the integral as $V = \int 2\pi\,(x)(\text{height})\,dx$, where x is the radius from the y-axis to the shell, and the height is the length of the region at x. Integrate over the x-values covering the region.

What information do I need to determine whether to use the disk or shell method for a given problem?

You need the region's boundaries, the axis of rotation, and which variable (x or y) simplifies the description of the region. Consider which method leads to easier integrals based on the region's orientation and limits.

Can the shell method be used for solids rotated around the x-axis?

Yes, the shell method can be used for rotation around the x-axis by integrating with respect to y, where the shells are oriented horizontally. Choose the method that simplifies the integral based on the problem's geometry.

Additional Resources

Use The Disk Method Or The Shell Method To Find The Volume Of The Solid Generated By Revolving The Region

In the realm of calculus, one of the fundamental challenges involves determining the volume of a three-dimensional object generated by revolving a two-dimensional region around an axis. This problem, while seemingly straightforward, demands precise methods and a profound understanding of the geometric and algebraic principles involved. Among the most effective techniques are the Disk Method and the Shell Method, each with unique applications and advantages. This comprehensive review explores these methods in depth, elucidating their principles, applications, and the nuanced distinctions that guide their selection.

Introduction to Volumetric Revolution of Regions

The process of generating solids of revolution is central to calculus, with applications spanning engineering, physics, and even biological modeling. Given a region bounded by curves in the xy-plane, revolving this region around an axis produces a three-dimensional solid. Calculating its volume is a classic problem that can be approached through integral calculus.

Two predominant approaches—the Disk/Washer Method and the Shell Method—are used depending on the region's shape, the axis of revolution, and the ease of setting up the integral. Both methods involve slicing the solid into infinitesimal elements, but they differ fundamentally in their conceptual framework and computational procedure.

Fundamental Concepts and Setup

Before delving into the specifics of each method, it is essential to understand the core idea:

- The goal: Find the volume V of the solid formed when a region R is revolved about a specified axis.
- The approach: Represent the volume as an integral of infinitesimal slices or shells, tailored to the geometry and axis of rotation.

The choice between the Disk and Shell methods hinges on factors such as the orientation of the region, the axis of revolution, and the simplicity of the integral setup.

The Disk Method: Principles and Applications

Overview and Concept

The Disk Method involves slicing the solid perpendicular to the axis of revolution, resulting in a series of disks or washers. Each disk's volume is computed as the area of its cross-section multiplied by its infinitesimal thickness.

When to Use the Disk Method

- When the cross-sections of the solid perpendicular to the axis of revolution are circular disks.
- When the region is easily described as a function of the variable perpendicular to the axis of revolution.
- Typically employed when revolving around the x-axis or y-axis, especially when the region is bounded by functions of the form $y = f(x)$ or $x = g(y)$.

Mathematical Formulation

Suppose the region is bounded between $y = f(x)$ and $y = g(x)$, with $a \leq x \leq b$, and revolved around the x-axis. The volume is:

$$
V = \pi \int_a^b \left[R_{\text{outer}}^2 - R_{\text{inner}}^2 \right] dx
$$

- R_{outer} and R_{inner} are the radii of the outer and inner disks, respectively.

- For a solid of revolution around the x-axis, $R =$ the distance from the x-axis to the curve y.

Example: Revolving a Region About the x-Axis

Suppose $y = f(x)$, with $f(x) \geq 0$, and the region between $x = a$ and $x = b$. The volume of the solid obtained by revolving this region about the x-axis is:

$$
V = \pi \int_a^b [f(x)]^2 dx
$$

Limitations

- Less effective when the region is better described with respect to the y-axis or when the axis of revolution is not horizontal.
- Can become cumbersome if the region has holes or multiple components, requiring washers instead of simple disks.

The Shell Method: Principles and Applications

Overview and Concept

The Shell Method employs cylindrical shells formed by slicing the region parallel to the axis of revolution. Each shell's volume is computed as the lateral surface area times its thickness.

When to Use the Shell Method

- When the region is more naturally described with respect to the variable parallel to the axis of revolution.
- When revolving around vertical axes (e.g., y-axis) and the region's boundary functions are expressed as $y = f(x)$.
- When the shell method simplifies the integral setup compared to the disk method.

Mathematical Formulation

Suppose the region is bounded between $x=a$ and $x=b$, and you revolve it around the y-axis. The volume is:

$$
V = 2\pi \int_a^b x \cdot f(x) \, dx
$$

- Here, x is the radius of each cylindrical shell.
- $f(x)$ represents the height of the shell.

Example: Revolving a Region About the y-Axis

If the region between $y=0$ and $y=f(x)$, for $x \in [a, b]$, is revolved around the y-axis, the volume is:

$$
V = 2\pi \int_a^b x \cdot f(x) \, dx
$$

Limitations

- Less effective when the region is better described with respect to the x-axis.
- Can involve more complex algebra if functions are complicated.

Comparative Analysis: Disk vs. Shell Method

Criterion	Disk Method	Shell Method
Suitable for	Regions perpendicular to the axis of revolution	Regions parallel to the axis of revolution
Integral setup	Involves cross-sectional areas	Involves cylindrical shells
Geometric intuition	Slicing perpendicular to axis	Slicing parallel to axis
Common axes of revolution	Horizontal or vertical (x or y axes)	Vertical or horizontal axes, depending on problem structure
Complexity	Simpler when slices are disks; complex with washers	Often simpler with cylindrical shells

---

Practical Guidelines for Method Selection

- Revolve around the x-axis:
- Use the Disk Method if the region is described as $y = f(x)$.
- Use the Shell Method if the region is more naturally described as a function of y.

- Revolve around the y-axis:
- Use the Shell Method when the region is described in terms of x.
- Use the Disk Method if the region's boundary is easier to express as y-functions.

- Complex regions:

- Sometimes, combining methods or transforming the region (e.g., via a change of variables) simplifies the calculation.

Step-by-Step Example: Volume of a Revolved Region

Problem Statement

Calculate the volume of the solid generated by revolving the region bounded by:

- $y = x^2$
- $y = 4$
- $x = 0$

about the y-axis.

Solution Using the Shell Method

1. Identify the region:

- The region between $y = x^2$ and $y=4$, from $x=0$ to $x=2$ (since $x^2=4 \Rightarrow x=2$).

2. Set up the integral:

- Since revolving around the y-axis and the region described in terms of x, the shell method is more straightforward.

3. Compute the volume:

$$
V = 2\pi \int_{x=0}^{x=2} x \cdot f(x) \, dx = 2\pi \int_0^2 x \cdot 4 \, dx
$$

(since the height of each shell is from $y=x^2$ up to $y=4$, but in the shell method, the radius is x and height is the difference in y values):

Alternatively, considering the shell's height as the difference in y, for shells at x:

$$
V = 2\pi \int_0^{2} x \cdot (4 - x^2) dx
$$

4. Evaluate the integral:

$$
V = 2\pi \int_0^2 x(4 - x^2) dx = 2\pi \int_0^2 (4x - x^3) dx
$$

$$
V = 2\pi \left[2x^2 - \frac{x^4}{4} \right]_0^2 = 2\pi \left[2 \cdot 4 - \frac{16}{4} \right] = 2\pi (8 - 4) = 2\pi \times 4 = 8\pi
$$

Result: The volume of the solid is 8π.

Advanced Considerations

Multiple Regions and Composite Solids

In complex scenarios involving multiple regions or non-standard boundaries, it may be necessary to:

- Split the region into simpler parts.
- Apply different methods to each part.
- Sum the volumes to obtain the total.

Using Both Methods in Tandem

Sometimes, the most efficient approach involves applying the Disk Method in one part of the problem and the Shell Method in another, depending on the axis and region orientation.

Numerical and Computational Methods

Use The Disk Method Or The Shell Method To Find The Volume Of The Solid Generated By Revolving The Region

Use The Disk Method Or The Shell Method To Find The Volume Of The Solid Generated By Revolving The Region

Related Articles

- The Log Ride At An Amusement Park Has A Single Hill With An Incline Of 127 Feet And A

Decline Of 93 Feet.
- 1. Magnesium Sulfate ($MgSO_4$), Also Known As Epsom Salt, Is A Common Ingredient In Bathing Salts. A Typical
- Why Did The Leaders Of The Newly Independent United States Dislike The Idea Of Letting Ordinary Americans

Back to Home