# Use The Fourier Transform To Derive The Poisson Integral Formula For The Following Boundary Value Problem

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Introduction

Solving boundary value problems (BVPs) for Laplace's equation is a fundamental task in mathematical physics, engineering, and applied mathematics. Among the classical solutions, the Poisson integral formula stands out as a powerful tool to solve Dirichlet problems in the unit disk. The derivation of this formula can be approached via various methods, including complex analysis, potential theory, and integral transforms. In this article, we will explore how the Fourier transform can be employed to derive the Poisson integral formula for a specific boundary value problem, elucidating the underlying techniques and their significance.

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Background and Motivation

The Boundary Value Problem

Consider the Laplace equation in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$:

$$
\begin{cases}
\Delta u(r, \theta) = 0, & 0 \leq r < 1, \quad 0 \leq \theta < 2\pi, \\
u(1, \theta) = f(\theta), & 0 \leq \theta < 2\pi,
\end{cases}
$$

where $f(\theta)$ is a given continuous function defined on the boundary $r = 1$. The goal is to find the harmonic function $u(r, \theta)$ inside the disk that matches the boundary data.

Significance of the Poisson Integral Formula

The Poisson integral formula provides an explicit solution:

$$
u(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} P(r, \theta - \phi) f(\phi) \, d\phi,
$$

where $P(r, \theta)$ is the Poisson kernel:

$$
P(r, \theta) = \frac{1 - r^2}{1 - 2r \cos \theta + r^2}.
$$

This formula elegantly reconstructs harmonic functions from boundary data and is essential in potential theory, harmonic analysis, and various applied fields.

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## The Fourier Transform: An Overview

### Definition and Basic Properties

The Fourier transform $\mathcal{F}$ of a function $g(\theta)$ defined on $\mathbb{R}$ (or $[0, 2\pi)$ with periodic extension) is:

$$
\hat{g}(n) = \int_{0}^{2\pi} g(\theta) e^{-i n \theta} \, d\theta,
$$

with the inverse transform:

$$
g(\theta) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{g}(n) e^{i n \theta}.
$$

In the context of functions on the circle, Fourier series are closely related to Fourier transforms.

### Relevance to Boundary Value Problems

The Fourier transform simplifies convolution operations, transforms differential equations into algebraic equations, and enables the explicit solution of PDEs with periodic boundary conditions. When applied to the boundary data $f(\theta)$, it facilitates the derivation of integral formulas such as the Poisson kernel.

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## Deriving the Poisson Integral Formula Using Fourier Transform

### Step 1: Expressing the Boundary Data via Fourier Series

Given the boundary function $f(\theta)$, expand it as a Fourier series:

$$
f(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{i n \theta},
$$

where the Fourier coefficients $c_n$ are:

$$
c_n = \frac{1}{2\pi} \int_{0}^{2\pi} f(\phi) e^{-i n \phi} \, d\phi.
$$

This expansion leverages the periodicity and facilitates the application of Fourier analysis tools.

Step 2: Recognizing the Harmonic Extension

The harmonic extension $u(r, \theta)$ inside the disk can be represented as a sum of harmonic functions:

$$
u(r, \theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{i n \theta}.
$$

This is a classical solution obtained by solving Laplace's equation in polar coordinates, using separation of variables.

Step 3: Applying the Fourier Transform to the Interior Solution

The key idea is to interpret the coefficients $c_n r^{|n|}$ as the Fourier coefficients of $u(r, \theta)$:

$$
u(r, \theta) = \sum_{n=-\infty}^{\infty} c_n r^{|n|} e^{i n \theta}.
$$

Since $c_n$ are obtained from the boundary data $f$, the boundary condition at $r=1$ yields:

$$
u(1, \theta) = \sum_{n=-\infty}^{\infty} c_n e^{i n \theta} = f(\theta).
$$

Step 4: Expressing $u(r, \theta)$ via Convolution

Using the inverse Fourier series representation, the solution inside the disk can be written as the convolution of $f$ with the Poisson kernel:

$$
u(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} P(r, \theta - \phi) f(\phi) \, d\phi,
$$

where the kernel $P(r, \theta)$ is obtained by summing the Fourier series:

$$
P(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{i n \theta}.
$$

Step 5: Summation of the Fourier Series for the Poisson Kernel

The sum

$$
\sum_{n=-\infty}^{\infty} r^{|n|} e^{i n \theta}
$$

can be explicitly computed. Recognizing it as a geometric series, we find:

$$
P(r, \theta) = \frac{1 - r^2}{1 - 2 r \cos \theta + r^2}.
$$

This is the classical Poisson kernel, which acts as a mollifier, smoothing the boundary data into a harmonic function inside the disk.

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### The Final Poisson Integral Formula

By synthesizing the above steps, the harmonic function $u(r, \theta)$ that solves the BVP can be expressed explicitly as:

$$
\boxed{
u(r, \theta) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{1 - r^2}{1 - 2 r \cos(\theta - \phi) + r^2} f(\phi) \, d\phi.
}
$$

This integral formula reconstructs the harmonic function $u$ in the disk from its boundary values $f(\phi)$, demonstrating the power of the Fourier transform approach in deriving integral solutions for PDEs.

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## Significance and Applications

### Advantages of Fourier Transform Method

- Explicit Solutions: Provides a clear pathway from boundary data to interior harmonic functions.
- Analytic Clarity: Illuminates the spectral nature of harmonic functions and their boundary behavior.
- Computational Utility: Facilitates numerical approximation of solutions via Fourier series.

### Broader Applications

- Potential Theory: Solving Laplace's equation in various geometries.
- Signal Processing: Analyzing periodic signals and their harmonic components.
- Electromagnetism & Fluid Dynamics: Modeling potential fields with prescribed boundary

conditions.
- Mathematical Physics: Studying steady-state temperature distributions, electrostatics, and more.

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Conclusion

The Fourier transform serves as a fundamental tool in the derivation of the Poisson integral formula for the classical Dirichlet problem in the unit disk. By expanding boundary data into Fourier series, recognizing the harmonic extension as a Fourier series with specific coefficients, and summing these series explicitly, one arrives at the Poisson kernel and its integral representation. This approach not only simplifies the derivation but also deepens the understanding of the spectral nature of harmonic functions, underscoring the profound connection between harmonic analysis and partial differential equations.

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# Frequently Asked Questions

## What is the main goal of deriving the Poisson integral formula using the Fourier transform?

The main goal is to find a solution to the Laplace equation in a disk with specified boundary conditions by expressing it as an integral involving the boundary data, leveraging the Fourier transform to facilitate this derivation.

## How does the Fourier transform simplify the process of solving boundary value problems like the Poisson equation?

The Fourier transform converts differential equations into algebraic equations in the frequency domain, making it easier to handle boundary conditions and solve for the potential function in problems like the Poisson equation.

## What boundary conditions are typically involved in deriving the Poisson integral formula for a disk?

The boundary condition usually specifies the function's values on the boundary circle of the disk, such as Dirichlet boundary conditions where the function's values are given along the boundary.

## Can you explain the role of the Fourier series in deriving the Poisson integral formula?

Fourier series decompose the boundary data into angular modes, allowing the solution to be expressed as a sum (or integral) over these modes, which leads directly to the integral representation known as the Poisson integral formula.

## How is the Fourier transform applied to boundary data in the context of the Poisson integral formula?

The boundary data function is Fourier transformed with respect to the angular variable, transforming the boundary condition into a form that can be extended into the interior via the Poisson kernel derived from the Fourier components.

## What is the significance of the Poisson kernel in the Fourier transform derivation of the Poisson integral formula?

The Poisson kernel acts as the fundamental solution that, when integrated against boundary data, produces the harmonic extension inside the disk; it naturally emerges from the Fourier mode solutions and the transform process.

## How does the Fourier transform relate to the harmonicity of solutions in the boundary value problem?

Since the Fourier transform converts the Laplace equation into an algebraic form in the frequency domain, it confirms that the solutions are harmonic functions, and facilitates deriving explicit integral formulas like the Poisson integral.

## In what way does the Fourier transform approach generalize to other boundary value problems beyond the disk?

The Fourier transform technique provides a systematic way to handle problems with periodic or symmetric boundary conditions, and can be extended to other geometries and PDEs where similar spectral methods apply.

## What are the key steps involved in using the Fourier transform to derive the Poisson integral formula for the boundary value problem?

The key steps include: (1) applying the Fourier transform to the boundary data, (2) solving the transformed PDE to find the harmonic extension in the frequency domain, (3) identifying the Poisson kernel as the inverse transform component, and (4) integrating the boundary data against this kernel to obtain the integral formula.

# Additional Resources

Use The Fourier Transform To Derive The Poisson Integral Formula For The Following Boundary Value Problem

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In the realm of mathematical physics and engineering, solving boundary value problems (BVPs) is a fundamental task. These problems often involve finding a function that satisfies a differential equation within a domain and adheres to specified conditions on the boundary. One classic example is Laplace's equation in the upper half-plane, where the goal is to determine a harmonic function given its boundary values. The Poisson integral formula provides a powerful tool for such problems, enabling the explicit construction of solutions from boundary data.

While classical methods such as separation of variables and conformal mappings are well-known, the Fourier transform offers an elegant and versatile approach to deriving these solutions. By transforming the differential equation into an algebraic equation in the frequency domain, we can simplify the problem considerably and then invert the transform to obtain the solution in the original domain. This article explores how the Fourier transform can be employed to derive the Poisson integral formula for a boundary value problem involving Laplace's equation in the upper half-plane, illuminating both the methodology and its underlying principles.

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Understanding the Boundary Value Problem

Before diving into the Fourier transform approach, it's critical to understand the specific boundary value problem we are addressing.

The Problem Statement:

Find a function $u(x, y)$, harmonic in the upper half-plane $y > 0$, such that:

1. Laplace's Equation:
$$
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad y > 0
$$
2. Boundary Condition:

$$
u(x, 0) = f(x), \quad \text{for all } x \in \mathbb{R}
$$

The goal is to find an explicit formula for $u(x, y)$ in terms of the boundary data $f(x)$.

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The Rationale for Using the Fourier Transform

The Fourier transform is a potent tool for handling linear partial differential equations, especially when dealing with problems defined on unbounded domains like the real line. Its ability to convert derivatives into algebraic multipliers simplifies the differential equation to an algebraic equation in the frequency domain.

Key reasons for employing the Fourier transform include:

- Simplification of derivatives: Differentiation in the spatial domain translates into multiplication by $i \xi$ in the frequency domain.
- Handling boundary conditions: The transform makes it straightforward to incorporate boundary data, especially when the boundary is at a fixed plane like $y=0$.
- Inversion to obtain explicit solutions: The inverse Fourier transform then reconstructs the solution in the spatial domain, often leading to integral formulas like the Poisson integral.

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Applying the Fourier Transform to the Boundary Value Problem

Step 1: Fixing the Boundary Plane and Transforming in $x$

Given the problem's symmetry along the $x$-axis, we perform the Fourier transform with respect to $x$. For a function $u(x, y)$, define:

$$
\hat{u}(\xi, y) = \mathcal{F}_x\{u(x, y)\} = \int_{-\infty}^\infty u(x, y) e^{-i \xi x} \, dx
$$

Similarly, the boundary data:

$$
\hat{f}(\xi) = \mathcal{F}_x\{f(x)\} = \int_{-\infty}^\infty f(x) e^{-i \xi x} \, dx
$$

Applying the Fourier transform to Laplace’s equation yields:

$$
\mathcal{F}_x\left\{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right\} = 0
$$

which simplifies to:

$$
-(i \xi)^2 \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} = 0
$$

or equivalently,

$$
\frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} - \xi^2 \hat{u}(\xi, y) = 0
$$

This is a second-order ordinary differential equation (ODE) in $y$ for each fixed $\xi$.

Step 2: Solving the ODE in $y$

The general solution to this ODE involves exponential functions:

$$
\hat{u}(\xi, y) = A(\xi) e^{|\xi| y} + B(\xi) e^{-|\xi| y}
$$

Since the solution must be harmonic in the upper half-plane $y > 0$ and physically reasonable (bounded as $y \to \infty$), we discard the exponentially growing term $e^{|\xi| y}$. This leaves:

$$
\hat{u}(\xi, y) = C(\xi) e^{-|\xi| y}
$$

where $C(\xi)$ is determined by the boundary condition at $y=0$:

$$
u(x, 0) = f(x) \implies \hat{u}(\xi, 0) = \hat{f}(\xi)
$$

Therefore,

$$
\hat{u}(\xi, 0) = C(\xi) = \hat{f}(\xi)
$$

and the solution in the frequency domain becomes:

$$
\hat{u}(\xi, y) = \hat{f}(\xi) e^{-|\xi| y}
$$

---

Inverse Fourier Transform and Derivation of the Poisson Kernel

Having solved for $\hat{u}(\xi, y)$, the next step involves retrieving $u(x, y)$ via the inverse Fourier transform:

$$
u(x, y) = \frac{1}{2 \pi} \int_{-\infty}^\infty \hat{u}(\xi, y) e^{i \xi x} \, d\xi
$$

Substituting the expression for $\hat{u}(\xi, y)$:

$$
u(x, y) = \frac{1}{2 \pi} \int_{-\infty}^\infty \hat{f}(\xi) e^{-|\xi| y} e^{i \xi x} \, d\xi
$$

Recognizing this as a convolution in the spatial domain, and applying the Fourier inversion theorem, leads to an integral representation:

$$
u(x, y) = \int_{-\infty}^\infty f(t) P_y(x - t) \, dt
$$

where $P_y(x)$ is known as the Poisson kernel in the upper half-plane:

$$
P_y(x) = \frac{1}{2 \pi} \int_{-\infty}^\infty e^{-|\xi| y} e^{i \xi x} \, d\xi
$$

---

## Computing the Poisson Kernel: The Final Integral Formula

The integral defining $P_y(x)$ can be evaluated explicitly. Recognizing the Fourier transform of the Laplace distribution, the integral simplifies to:

$$
P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}
$$

which is the classical Poisson kernel for the upper half-plane.

Thus, the solution to the boundary value problem is:

$$
u(x, y) = \int_{-\infty}^\infty f(t) \frac{1}{\pi} \frac{y}{(x - t)^2 + y^2} \, dt
$$

This is the Poisson integral formula, expressing the harmonic extension of the boundary data $f$ into the upper half-plane.

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## Significance and Broader Implications

The derivation of the Poisson integral formula using Fourier transforms not only provides an elegant solution methodology but also highlights the profound connection between harmonic functions, integral transforms, and complex analysis. It demonstrates how transforming a boundary value problem into the frequency domain simplifies the differential equation into an algebraic form, which is easier to solve.

This approach is widely applicable beyond the upper half-plane, extending to various domains and higher-dimensional problems. In particular, the Fourier transform technique underpins many advanced methods in potential theory, signal processing, and quantum mechanics. It also offers insight into the nature of harmonic functions, illustrating how boundary data propagate into the domain through a convolution with a kernel that encodes the geometry of the problem.

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Conclusion

The Fourier transform serves as a powerful bridge between differential equations and algebraic equations in the frequency domain. Its application to boundary value problems, such as the Laplace equation in the upper half-plane, yields the Poisson integral formula—a fundamental result in harmonic analysis. This derivation underscores the versatility of Fourier analysis in solving classical PDEs, providing explicit, integral representations of solutions that are both mathematically elegant and practically useful.

By understanding this process, mathematicians and engineers can better appreciate the interconnectedness of analytical techniques and develop more sophisticated tools for tackling complex boundary problems across various scientific disciplines.

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