Write The Function In Terms Of Unit Step Functions. Find The Laplace Transform Of The Given Function.f(t)

Write The Function In Terms Of Unit Step Functions. Find The Laplace Transform Of The Given Function.f(t) is a fundamental problem in the realm of applied mathematics, especially within the fields of engineering, signal processing, and systems analysis. By expressing a piecewise or complex time-domain function in terms of unit step functions, we simplify the process of analyzing and transforming the function, particularly when applying the Laplace transform. This approach not only makes the mathematical operations more straightforward but also enhances our understanding of how various signals and systems behave over time.

In this comprehensive guide, we will explore the methodology of rewriting functions using unit step functions, demonstrate how to find the Laplace transform of such functions, and provide practical examples to solidify these concepts. Whether you are a student, a researcher, or a professional working with dynamic systems, mastering this technique is crucial for efficient analysis and problem-solving.

Understanding the Unit Step Function

Before delving into the process of rewriting functions, it’s essential to understand what the unit step function is and why it’s so useful.

Definition of the Unit Step Function

The unit step function, often denoted as $u(t)$, is a mathematical function defined as:

$$
u(t) = \begin{cases}
0, & t < 0 \\
1, & t \geq 0
\end{cases}
$$

This function essentially "steps" from 0 to 1 at $t=0$. It acts as a switch that can turn on or off parts of a function at specific points in time.

Properties of the Unit Step Function

Understanding these properties helps when manipulating functions:

- Shifting Property: $u(t - a)$ shifts the step to $t = a$, turning on at that point.
- Scaling: The function can be scaled by constants, e.g., $A \cdot u(t - a)$.
- Multiplication with other functions: When multiplying a function $f(t)$ by $u(t - a)$, the effect is to "turn on" $f(t)$ at $t = a$.

Expressing Functions in Terms of Unit Step Functions

Many functions encountered in practice are piecewise-defined or involve different behaviors over different intervals. Expressing these functions as combinations of shifted unit step functions allows for a unified representation, simplifying the application of transforms like the Laplace transform.

Methodology for Rewriting Functions

The general approach involves:

1. Identify the intervals where the function changes behavior.
2. Express each segment as a constant or a simple function.
3. Use the unit step function to turn "on" each segment at the appropriate time.
4. Sum all segments to form the complete expression.

Example of Rewriting a Piecewise Function

Suppose $f(t)$ is defined as:

$$
f(t) = \begin{cases}
0, & 0 \leq t < 2 \\
3, & t \geq 2
\end{cases}
$$

This can be expressed as:

$$
f(t) = 3 \cdot u(t - 2)
$$

because $u(t - 2)$ equals 0 for $t < 2$ and 1 for $t \geq 2$, turning on the value 3 at $t=2$.

Formulating the Function in Terms of Unit Step Functions

Let's consider a more complex example to illustrate the process.

Example: A Piecewise Function

Suppose the function $f(t)$ is:

$$
f(t) = \begin{cases}
t, & 0 \leq t < 1 \\
2t + 1, & 1 \leq t < 3 \\
0, & t \geq 3
\end{cases}
$$

Step 1: Express each segment

- For $0 \leq t < 1$: $f(t) = t$
- For $1 \leq t < 3$: $f(t) = 2t + 1$
- For $t \geq 3$: $f(t) = 0$

Step 2: Use unit step functions to "turn on" each segment

The function can be written as:

$$
f(t) = t \cdot [u(t) - u(t - 1)] + (2t + 1) \cdot [u(t - 1) - u(t - 3)]
$$

Breaking down:

- $t \cdot [u(t) - u(t - 1)]$: active between 0 and 1.
- $(2t + 1) \cdot [u(t - 1) - u(t - 3)]$: active between 1 and 3.
- Outside these, the function is zero.

Step 3: Final expression

$$
f(t) = t \cdot [u(t) - u(t - 1)] + (2t + 1) \cdot [u(t - 1) - u(t - 3)]
$$

This representation simplifies the process of applying the Laplace transform.

Applying the Laplace Transform to Functions Using Unit Step Functions

Once the function is expressed in terms of unit step functions, the next step is to find its Laplace transform.

Laplacian Properties for Unit Step Functions

The Laplace transform of the unit step function shifted by a is:

$$
\mathcal{L}\{ u(t - a) \} = \frac{e^{-a s}}{s}
$$

For functions multiplied by $u(t - a)$, the Laplace transform involves a shift property:

$$
\mathcal{L}\{ f(t) \cdot u(t - a) \} = e^{-a s} \cdot \mathcal{L}\{ f(t + a) \} \quad \text{(with appropriate adjustments)}
$$

More generally, for a function expressed as a sum of terms involving shifted step functions, the Laplace transform can be obtained by applying linearity and the shift property.

Calculating the Laplace Transform: Step-by-step

1. Express the function in terms of $u(t - a)$.
2. Apply the Laplace transform to each term separately.
3. Use the shift property to handle the $u(t - a)$ terms.
4. Sum the results to obtain the overall Laplace transform.

Practical Examples

To solidify these concepts, let's go through a detailed example.

Example: Laplace Transform of a Piecewise Function

Given:

$$
f(t) = t \cdot u(t) + 4 \cdot u(t - 2)
$$

Step 1: Rewrite in terms of unit step functions

This function is already expressed in terms of $u(t)$ and $u(t - 2)$.

Step 2: Apply the Laplace transform

- For $t \cdot u(t)$:

$$
\mathcal{L}\{ t \} = \frac{1}{s^2}
$$

- For $4 \cdot u(t - 2)$:

$$
\mathcal{L}\{ u(t - 2) \} = \frac{e^{-2 s}}{s}
$$

Since this term is a constant times the step function, the Laplace transform is:

$$
4 \cdot \frac{e^{-2 s}}{s}
$$

Step 3: Combine the transforms

The total Laplace transform:

$$
F(s) = \frac{1}{s^2} + 4 \cdot \frac{e^{-2 s}}{s}
$$

This example demonstrates how expressing a function with unit step functions simplifies the Laplace transform process.

Summary and Key Takeaways

- The unit step function $u(t - a)$ is a powerful tool for representing piecewise functions and signals that turn on at specific times.
- Rewriting functions in terms of shifted unit step functions allows for a unified, algebraic expression, making the application of the Laplace transform straightforward.
- The Laplace transform of shifted unit step functions involves exponential factors $e^{-a s}$, reflecting the "delay" introduced by the shift.

- Practical problems often involve breaking down complex or piecewise functions into sums involving $u(t - a)$, then applying linearity and shift properties of the Laplace transform to evaluate.

Conclusion

Mastering the process of expressing functions in terms of unit step functions and then finding their Laplace transforms is a fundamental skill in systems analysis. It simplifies complex, piecewise, or delayed signals into manageable algebraic forms, facilitating analysis and solution of differential equations, control systems, and signal processing problems. Practice with various functions and their representations will deepen understanding and proficiency, enabling more efficient problem-solving in engineering and applied sciences.

Further Reading and Resources