Describe All Solutions Of Ax = 0 In Parametric Vector Form, Where A Is Row Equivalent To The Given Matrix.

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Understanding the solutions to the homogeneous system of linear equations $Ax = 0$ is fundamental in linear algebra. When matrix A is row equivalent to a given matrix, it implies that they share the same solution set. Expressing all solutions in parametric vector form provides a clear and comprehensive way to understand the structure of these solutions, especially when dealing with systems that have infinitely many solutions. This article explores how to describe all solutions of $Ax = 0$ in parametric vector form, emphasizing the importance of row equivalence and the steps involved in deriving the parametric form.

Introduction to Homogeneous Systems and Row Equivalence

What is a Homogeneous System?

A homogeneous system of linear equations takes the form:

$$
Ax = 0
$$

where:
- A is an $m \times n$ matrix,
- x is an $n \times 1$ column vector of variables,
- 0 is the zero vector of size $m \times 1$.

The key property of such systems is that they always have at least one solution—the trivial solution $x = 0$. The main question is whether there are infinitely many solutions or only the trivial one.

Row Equivalence and Its Significance

Two matrices A and B are said to be row equivalent if one can be obtained from the other through a finite sequence of elementary row operations. These operations include:

- Swapping two rows,
- Multiplying a row by a non-zero scalar,
- Adding a multiple of one row to another.

Row equivalence preserves the solution set of the associated homogeneous system because elementary row operations correspond to invertible row transformations. Therefore, the solution sets of $Ax = 0$ and $Bx = 0$ are the same if A and B are row equivalent.

Finding the RREF and Its Role in Solution Description

Reduced Row Echelon Form (RREF)

The process of solving $Ax = 0$ often involves reducing A to its Reduced Row Echelon Form (RREF). The RREF of a matrix has the following properties:

- Leading entries (pivots) are 1,
- Each leading 1 is the only non-zero entry in its column,
- Leading 1s move to the right as you move down rows,
- Rows with all zeros are at the bottom.

Reducing A to RREF simplifies the process of identifying pivot and free variables, which is crucial for parametric solution form.

Why is RREF Important?

- It reveals the pivot positions.
- It helps determine which variables are basic (pivot variables) and which are free.
- It makes it straightforward to write the general solution in parametric vector form.

Step-by-Step Process to Describe All Solutions in Parametric Vector Form

Step 1: Write the Matrix A and Find Its RREF

Begin with your matrix A. Use elementary row operations to reduce it to RREF. This process involves:

- Forward elimination to reach echelon form,
- Backward substitution to achieve reduced form.

Step 2: Identify Pivot and Free Variables

- Pivot variables: Correspond to columns with leading 1s in RREF.
- Free variables: Correspond to columns without pivots.

For example, suppose A is a 3×4 matrix and its RREF has leading 1s in columns 1 and 3. Then:

- x_1 and x_3 are pivot variables,
- x_2 and x_4 are free variables.

Step 3: Express Pivot Variables in Terms of Free Variables

Use the RREF equations to express each pivot variable as a linear combination of free variables.

For example, in a simplified scenario:

$$
x_1 = 2x_2 - x_4 \\
x_3 = x_2 + 3x_4
$$

where x_2 and x_4 are free parameters.

Step 4: Assign Parameters to Free Variables

Choose parameters (say t and s) for the free variables:

$$
x_2 = t, \quad x_4 = s
$$

with $t, s \in \mathbb{R}$.

Step 5: Write the General Solution as a Vector in Parametric Form

Substitute the parameters into the expressions for pivot variables to obtain the general solution:

$$
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
= t \begin{bmatrix}
2 \\
1 \\
1 \\
0
\end{bmatrix}
+ s \begin{bmatrix}
-1 \\
0 \\
3 \\
1
\end{bmatrix}
$$

This expresses the entire solution set as a linear combination of vectors scaled by free parameters.

Parametric Vector Form of the Solution Set

General Form

The solution to $Ax = 0$ can be expressed as:

$$
x = c_1 v_1 + c_2 v_2 + \cdots + c_k v_k
$$

where:
- $v_1, v_2, \ldots, v_k$ are the basic solutions corresponding to each free variable,
- $c_1, c_2, \ldots, c_k$ are arbitrary scalars (parameters).

In the context of the previous example:

$$
x = t \begin{bmatrix}
2 \\
1 \\
1 \\
0
\end{bmatrix}
+ s \begin{bmatrix}
-1 \\
0 \\
3 \\
1
\end{bmatrix}
\quad \text{where} \quad t, s \in \mathbb{R}
$$

This parametric vector form makes it easy to see the structure of the solution space.

Advantages of Parametric Vector Form

- Clearly shows the dimension of the solution space (number of free variables).
- Provides a basis for the null space of A.
- Facilitates understanding of the solution's geometric interpretation (e.g., lines, planes).

Applications and Examples

Example 1: Homogeneous System with Multiple Solutions

Given matrix:

$$
A = \begin{bmatrix}
1 & 2 & 0 & -1 \\
0 & 0 & 1 & 3
\end{bmatrix}
$$

Row reduce to RREF:

$$
\text{RREF}(A) = \begin{bmatrix}
1 & 2 & 0 & -1 \\
0 & 0 & 1 & 3
\end{bmatrix}
$$

The pivot columns are 1 and 3, so:

- x_1 and x_3 are basic variables,
- x_2 and x_4 are free variables.

Expressed equations:

$$
x_1 = -2x_2 + x_4 \\
x_3 = -3x_4
$$

Let:

$$
x_2 = t, \quad x_4 = s
$$

then the solution vector:

$$
x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = t \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix} + s \begin{bmatrix}
1 \\
0 \\
-3 \\
1
\end{bmatrix}
$$

This parametric vector form provides a complete description of the solution set.

Example 2: Null Space Basis

The vectors multiplying the free parameters form a basis of the null space of A. For the above example, the basis vectors are:

$$
v_1 = \begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}, \quad
v_2 = \begin{bmatrix}
1 \\
0 \\
-3 \\
1
\end{bmatrix}
$$

Any solution is a linear combination of these basis vectors, emphasizing the importance of parametric vector form in understanding the null space.

Conclusion

Describing all solutions of the homogeneous system \(Ax = 0\) in parametric vector form is a powerful technique in linear algebra. When \(A\) is row equivalent to a matrix in RREF, the process involves identifying pivot and free variables, expressing pivot variables in terms of free variables, and then writing the solution as a linear combination of parameterized vectors.

This approach not only simplifies understanding the structure of the solution space but also provides

Frequently Asked Questions

What is the general approach to finding all solutions of the homogeneous system Ax = 0 when A is row equivalent to a given matrix?

The general approach involves reducing A to its row echelon form to identify free variables, then expressing the solution vector x in terms of these free variables, leading to the parametric vector form of the solutions.

How does row equivalence to a matrix simplify finding the solutions of Ax = 0?

Row equivalence transforms A into a simpler, often reduced, form where the leading variables and free variables are evident, making it straightforward to write the solutions parametrically.

What is meant by the parametric vector form of the solutions to Ax = 0?

The parametric vector form expresses the solution set as a linear combination of vectors multiplied by free parameters, capturing all possible solutions in a compact, vector-based expression.

How do you determine the free variables when solving Ax = 0 in parametric form?

Free variables are those not leading in the row echelon form of A; they correspond to columns without leading ones and are assigned parameters in the parametric solution.

If A is row equivalent to a matrix with a certain row echelon form, how do you write the solution set in parametric vector form?

Identify pivot and free variables from the row echelon form, assign parameters to free variables, and express pivot variables in terms of these parameters, resulting in a solution set expressed as a sum of parameter vectors.

Why is understanding the solutions in parametric vector form important in linear algebra?

It provides a complete description of the solution space, reveals the dimension and basis of the null space, and aids in understanding the structure of solutions for linear systems.

Can the parametric vector form of solutions be used to find the basis of the null space of A?

Yes, the vectors multiplied by the parameters in the parametric solution form constitute a basis for the null space of A.

Additional Resources

Solutions of Ax = 0 in Parametric Vector Form When A Is Row Equivalent to a Given Matrix

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Understanding the solution space of linear systems is fundamental in linear algebra, especially when dealing with homogeneous equations of the form $Ax = 0$. This article provides a comprehensive exploration of all solutions to the system $Ax = 0$ in parametric vector form, particularly when matrix A is row equivalent to a given matrix. Through a detailed examination of concepts, processes, and illustrative examples, we aim to equip you with a robust understanding of this essential topic.

Introduction to Homogeneous Systems and Row Equivalence

Before delving into the parametric solutions, it is crucial to establish a solid foundation by understanding what homogeneous systems are and the significance of row equivalence.

Homogeneous Systems: An Overview

A homogeneous system is a linear system where all constant terms are zero, expressed as:

$$ Ax = 0 $$

where:

- A is an $m \times n$ matrix,
- x is an $n \times 1$ vector of variables,
- The zero vector is of size $m \times 1$.

Key properties:

- The trivial solution $x = 0$ always exists.
- The solution set forms a subspace of $\mathbb{R}^n$.
- The nature of solutions depends on the rank of A:
- If $\operatorname{rank}(A) = n$, the only solution is the trivial one.
- If $\operatorname{rank}(A) < n$, there are infinitely many solutions.

Row Equivalence and Its Role

Two matrices A and B are row equivalent if one can be obtained from the other through a sequence of elementary row operations (row swapping, scaling, or adding multiples of one row to another). This concept establishes an important connection:

- Row equivalent matrices have the same row space.
- They share the same solutions for the associated homogeneous system, meaning:

$$ Ax = 0 \text{ and } Bx = 0 $$

have identical solution sets when A and B are row equivalent.

Knowing that A is row equivalent to a matrix in reduced form (e.g., reduced row echelon form, RREF) simplifies the process of solving the homogeneous system.

Transforming the Matrix: From Given Matrix to Row Echelon Form

The key step in finding all solutions to $Ax=0$ involves converting A to a simpler form that reveals the structure of solutions.

Row Reduction Process

The process involves elementary row operations to produce:

- Row echelon form (REF): where all zero rows are at the bottom, and leading coefficients are to the right of those in the rows above.
- Reduced row echelon form (RREF): where leading coefficients are 1, and each leading 1 is the only non-zero entry in its column.

The goal is to simplify the matrix to identify pivot variables and free variables:

- Pivot variables: Correspond to columns with leading 1s.
- Free variables: Columns without leading 1s, which can take arbitrary parameter values.

Example: Row Reduction of a Given Matrix

Suppose we have:

$$
A = \begin{bmatrix}
1 & 2 & -1 & 4 \\
2 & 4 & -2 & 8 \\
3 & 6 & -3 & 12
\end{bmatrix}
$$

Applying row operations to reach RREF:

1. Use row 1 as pivot:

$$
\text{R}_2 \leftarrow \text{R}_2 - 2\text{R}_1 \implies
$$

\begin{bmatrix}
1 & 2 & -1 & 4 \\
0 & 0 & 0 & 0 \\
3 & 6 & -3 & 12
\end{bmatrix}
\]

2. Eliminate row 3:

\[
\text{R}_3 \leftarrow \text{R}_3 - 3\text{R}_1 \implies
\begin{bmatrix}
1 & 2 & -1 & 4 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Result: The matrix has one pivot in column 1, indicating \(x_1\) is a pivot variable, while \(x_2, x_3, x_4\) are free variables.

General Form of Solutions in Parametric Vector Form

Once the matrix is in row echelon form, the next step is to express the solution set parametrically.

Identifying Pivot and Free Variables

- The pivot variables are directly expressed in terms of free variables.
- The free variables serve as parameters (usually denoted as \(t, s, \ldots\)) that span the solution space.

Constructing the Parametric Solution

The general process involves:

1. Assigning parameters to free variables.
2. Solving the pivot equations for the pivot variables in terms of these parameters.

3. Writing the solution vector \(x\) as a linear combination of vectors multiplied by parameters.

Example:

Suppose the RREF of \(A\) yields:

\[
x_1 + 2x_2 - x_4 = 0 \\
x_3 \text{ is free} \\
x_4 \text{ is free}
\]

Then, expressing pivot variables:

\[
x_1 = -2x_2 + x_4 \\
x_3 = s \\
x_4 = t
\]

with \(s, t \in \mathbb{R}\). The parametric vector form becomes:

\[
x =
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
=
x_2
\begin{bmatrix}
-2 \\
1 \\
0 \\
0
\end{bmatrix}
+ t
\begin{bmatrix}
1 \\
0 \\
0 \\

1
\end{bmatrix}
+ s
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}
\]

where \(x_2\) can be expressed in terms of the parameters or set as a free parameter if desired.

Explicit Procedure to Find All Solutions of Ax = 0

This section provides a step-by-step guide to derive the parametric vector form of solutions:

Step 1: Reduce A to Row Echelon Form or RREF

- Use elementary row operations to simplify the matrix.
- Identify pivot columns and free columns.

Step 2: Assign Parameters to Free Variables

- For each free variable, assign a parameter (e.g., \(t, s, u\)).

Step 3: Express Pivot Variables in Terms of Parameters

- Solve the pivot equations to write pivot variables as linear combinations of free variables.

Step 4: Write the General Solution Vector

- Combine all expressions into a single vector.

- Express the solution as a linear combination of vectors multiplied by parameters (the free variables).

Step 5: Interpret the Solution Space

- The solution set is a subspace of $\mathbb{R}^n$ spanned by the parametric vectors.
- The dimension of the solution space equals the number of free variables.

Significance of the Parametric Vector Form

The parametric vector form offers several advantages:

- Clarity: Clearly shows the structure of the solution space.
- Basis Identification: The vectors multiplied by parameters form a basis for the null space (solution space).
- Computational Efficiency: Facilitates solving larger systems and understanding their properties.
- Geometric Interpretation: Visualizes the solution set as a subspace spanned by basis vectors.

Special Cases and Additional Considerations

While the general approach is straightforward, some special cases warrant attention:

Unique Solution (Trivial Solution Only)

- When $\operatorname{rank}(A) = n$, no free variables exist.
- The only solution is $x=0$.

Infinite Solutions

- When $\operatorname{rank}(A) < n$, free variables exist, leading to infinitely many solutions.
- The solution space is a subspace of dimension $n - \operatorname{rank}(A)$.

Particular Example Revisited

Suppose:

\[
A = \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & -1
\end{bmatrix}
\]

The homogeneous system:

\[
\begin{cases}
x_1 + 2x_3 = 0 \\
x_2 - x_3 = 0
\end{cases}
\]

Expressed parametrically:

\[
x_3 = t \

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